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ABSTRACT. Let $F \in W^{1,n}_{\text{loc}}(\Omega;\mathbb{R}^n)$ be a mapping with non negative Jacobian $J_F(x) = \det DF(x) \geq 0$ for a. e. x in a domain $\Omega \subset \mathbb{R}^n$. The *dilatation* of F is defined (almost everywhere in Ω) by the formula

$$K(x) = \frac{|DF(x)|^n}{J_F(x)}.$$

If K is bounded, the mapping F is said to be quasiregular. These are a generalization to higher dimensions of holomorphic functions. The theory of higher dimensional quasiregular mappings began with Reŝhetnyak's theorem [R], stating that they are continuous, discrete and open, if they are nonconstant.

In some problems appearing in the nonlinear elasticity models suggested in [B1-2], the boundedness condition for K is too restrictive. Typically we only have that K^p is integrable for some p. In two dimensions, Iwaniec and Šverák [IŠ] have shown that $K \in L^1_{\text{loc}}$ is enough to guarantee the conclusion of Reŝhetnyak's theorem. In this paper we consider the higher dimensional case $n \geq 3$, and extend Reŝhetnyak's theorem to the case $K \in L^p_{\text{loc}}$, where $p > n-1$. This is known to be false for $p < n-1$ and is not known in the case $p = n-1$.

We follow the footsteps of Reŝhetnyak's original proof, however our equations are no longer strictly elliptic. We develop a method to deal with badly degenerate elliptic equations based on monotone functions estimates, that allows us to establish a weak Harnack's inequality for $\log(1/|F|)$. A nontrivial matter here, is the construction of appropriate test functions. We use a computer to exhibit an explicit smooth n-superharmonic "bump function" which approximates $\log(1/|x|)$.

1. Introduction. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a domain and $F\colon \Omega \to \mathbb{R}^n$ be a mapping in the Sobolev space $W^{1,n}_{\text{loc}}(\Omega; R^n)$ of functions in $L^n_{\text{loc}}(\Omega; R^n)$ whose

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distributional derivatives belong to $L^n_{\text{loc}}(\Omega; R^n)$. We can think of F as a deformation of some material whose initial configuration is Ω, and we seek some functional $I(F)$ representing the (nonlinear) elastic energy whose minimum is attained at F, see [B1-3] and [S]. The differential of F at a point x is denoted by $DF(x)$, its norm is

$$|DF(x)| = \sup\{|DF(x)h| : h \in \mathbb{R}^n,\ |h| = 1\}$$

and its Jacobian determinant is $J_F(x) = \det DF(x)$. We assume that F is orientation preserving, meaning that $J_F(x) \geq 0$ for a. e. $x \in \Omega$. The *dilatation* of F at the point x is defined by the ratio

$$K(x) = \frac{|DF(x)|^n}{J_F(x)}.$$

If $K(x) \in L^\infty(\Omega; R^n)$, then F is said to be a quasiregular mapping.

We will say that F is a mapping of *finite dilatation* if

$$1 \leq K(x) < \infty \quad \text{for a. e. } x \in \Omega;$$

that is, except for a set of measure zero in Ω, if $J_F(x) = 0$ then $DF(x) = 0$.

A basic result in the theory of quasiregular mappings [Re], states that they are either discrete and open or constant. In this paper, we consider the same problem for mappings with *finite dilatation.* Vodopyanov and Goldstein [VG], proved that mapping of *finite dilatation* are continuous and have monotone components. See [M] for a simple proof of these facts. An example of Ball [B2] shows that there are mappings satisfying $K \in L^p$ for every $p < n-1$ that fail to be discrete. From now on we will assume that $K \in L^p$ for some $p \geq n-1$.

A beautiful theorem of Iwaniec and Šverák [IS], shows that a mapping in the plane with integrable dilatation, $K(x) \in L^1$, can be expressed as an analytic function composed with a homeomorphism. In particular it must be open and discrete. The proof is based on the linear two dimensional Beltrami equation and does not generalize in an obvious way to higher dimensions.

Our main goal in this paper is to prove the following result:

Theorem 1. *Let $F \in W^{1,n}_{\text{loc}}(\Omega, R^n)$ be a nonconstant mapping whose dilatation $K(x)$ is in $L^p_{\text{loc}}(\Omega)$. Then, if $p > n-1$, the mapping F is continuous, discrete and open.*

Heinonen and Koskela [HK], have recently proved this theorem when F is quasi-light (i. e. for all $y \in \mathbb{R}^n$, $F^{-1}(y)$ is compact), and $K \in L^{n-1+\varepsilon}_{\text{loc}}(\Omega)$ for some positive ε, or assuming that $F \in W^{1,q}_{\text{loc}}(\Omega, R^n)$, where $q \geq n + 1/(n-2)$. Our approach is, however, different from theirs.

An announcement of the results in this paper appeared in the Bulletin of the American Mathematical Society [MV].

2. Degenerate elliptic equations and systems associated with F. Denote by $\operatorname{adj}(DF(x))$ the adjugate matrix of $DF(x)$ defined by the relation

$$DF(x)\cdot \operatorname{adj}(DF(x)) = J_F(x)\cdot I_n.$$

If $J_F(x)\neq 0$, we have $\operatorname{adj}((DF(x))) = J_F(x)(DF(x))^{-1}$ and, in general, the entries of $\operatorname{adj}(DF(x))$ are homogeneous polynomials of degree $(n-1)$ with respect to the variables $\partial F^i/\partial x_j$.

It is a well known fact that $\operatorname{adj}(DF)$ is divergence free. Indeed we have

$$\operatorname{div}\big(\operatorname{adj}(DF)(V\circ F)\big) = 0 \tag{2.1}$$

in the sense of distributions, where V is a C^1 vector field such that $\operatorname{div}V = 0$ (see for example [BI]).

Define

$$G(x) = \begin{cases} J_F(x)^{2/n}(DF(x)^t DF(x))^{-1} & \text{if } J_F(x) > 0\\ I & \text{otherwise.}\end{cases} \tag{2.2}$$

The matrix $G(x)$ is symmetric, has determinant 1 and measures how far $DF(x)$ is from being conformal. In fact the pull back of the euclidean metric under the mapping F is given in local coordinates by the matrix $DF(x)^t DF(x)$ whose coefficients are not necessarily bounded. A metric conformal to this one whose eigenvalues are controlled by $K(x)$ is obtained by multiplying by an appropriate power of the Jacobian $J_F(x)$. Expressing these metrics in local coordinates we get $G^{-1}(x)$. From the definition of dilatation we have the following estimate

$$\frac{1}{c_n(K(x))^{2/n}}|\xi|^2 \le \langle G(x)\xi\,,\,\xi\rangle \le c_n K^{2-2/n}(x)|\xi|^2, \tag{2.3}$$

where c_n is a constant depending only on the dimension n.

Let φ be a real valued function defined in $\mathbb{R}^n$ and consider $u=\varphi\circ F$. The chain rule gives $\nabla u(x) = DF(x)^t\nabla\varphi(x)$. A basic calculation that expresses the conformal invariance of the n dimensional Dirichlet integral is

$$\begin{aligned}&\langle G(x)\nabla u(x)\,,\,\nabla u(x)\rangle^{n/2-1}G(x)\nabla u(x)\\ &\qquad = \operatorname{adj}DF(x)|\nabla\varphi(F(x))|^{n-2}\nabla\varphi(F(x)).\end{aligned} \tag{2.4}$$

We see by applying (2.1) and (2.4) that if φ is n-harmonic, that is

$$\operatorname{div}(|\nabla\varphi(x)|^{n-2}\nabla\varphi(x)) = 0,$$

then u is *formally* a solution of

$$\operatorname{div}(A(x,\nabla u)) = 0, \tag{2.5}$$

where

$$A(x,\xi) = \langle G(x)\xi\,,\,\xi\rangle^{n/2-1}G(x)\xi$$

satisfies

$$\frac{1}{c_n K(x)}|\xi|^n \leq A(x,\xi)\cdot\xi \leq c_n K^{n-1}(x)|\xi|^n.$$

If K is bounded (F is quasiregular), the equations so obtained are of n-Laplacian type, and the known regularity theory, see [Se], gives right away the Hölder continuity of F and the Harnack inequality for $\log(1/|F|)$. Indeed, by taking $\varphi(y) = y_i$, $i = 1,\ldots,n$ we obtain that the i-th component of F, F^i is a solution of (2.5) and therefore Hölder continuous. By taking

$$\varphi(y) = \log\frac{1}{|y-b|}$$

we obtain that $\log(1/|F(x)-b|)$ is a solution of (2.5) in the open set $\Omega\setminus F^{-1}(b)$ and is in fact a nonnegative supersolution everywhere. A weak form of the Harnack inequality, see [BI], implies that $\log(1/|F(x)-b|)$ is in $W^{1,n-\varepsilon}_{\mathrm{loc}}$ for any ε positive. Standard results from nonlinear potential theory, see [HKM], give that $F^{-1}(b)$ is a set of n-capacity zero, and therefore it has Hausdorff dimension zero. It follows that $F^{-1}(b)$ can not contain an arc. Thus, it is totally disconnected making F a continuous sense preserving light mapping. A technical point must be discussed here. The mapping F must have positive topological index (sense preserving in the topological sense). That this is indeed the case for every finite dilatation mapping is observed in [HK]. Since continuous sense preserving light mappings are open and discrete, see [TY], we conclude that quasiregular mappings are open and discrete.

If K is only in L^p, it is not clear that we can use u, or a function of u, as a test function in the weak formulation of (2.5). In order to overcome this difficulty we use a more general version of (2.1) when V is not necessarily divergence free.

Proposition 1. *Let F be in $W^{1,n}(\Omega;\mathbb{R}^n)$ and V a C^1-vector field. Then*

$$\operatorname{div}((\operatorname{adj} DF)V\circ F) = [(\operatorname{div} V)\circ F]J_F \tag{2.6}$$

in the sense of distributions.

Remark. This formula is quite interesting in itself. It is trivial to check when both F and V are smooth. If we assume a priori that F is quasiregular, it holds for $V \in L^{n/(n-1)}$ as proved in [DS]. For general $F \in W^{1,n}$ it can be proved by an approximation argument along the lines of the proof of a similar statement found in [S], and for even more general F it can be found on [MTY].

Let $\varphi \in C_0^\infty(\Omega)$, the weak form of (2.6) is,

$$\int_\Omega \langle(\operatorname{adj} DF)V\circ F\,,\,\nabla\varphi\rangle\,dx = -\int_\Omega ((\operatorname{div} V)\circ F)(x)\varphi(x)J_F(x)\,dx. \tag{2.7}$$

Note that $\operatorname{adj} DF \in L^{n/(n-1)}_{\mathrm{loc}}(\Omega)$ and $J_F \in L^1_{\mathrm{loc}}(\Omega)$. By an approximation argument (2.7) also holds for any $\varphi \in W^{1,n}(\Omega) \cap L^\infty(\Omega)$ with compact support.

The key idea in our argument is to realize that equation (2.7) is like an elliptic partial differential equation for test functions of the form $\Phi(F(x))$. Explicitly, let $\eta \in C_0^\infty(G)$, $\eta \geq 0$, where G is a relatively compact domain in Ω. Let $\Phi \in C^2(\Omega')$, $\Phi \geq \delta$ for some positive δ. Choosing $\varphi(x) = \eta^n(x)\Phi^m(F(x))$, where $m < 0$ and using the chain rule we get

$$\begin{aligned}\nabla\varphi(x) &= n\eta^{n-1}(x)\Phi^m(F(x))\nabla\eta(x)\\ &\quad + m\eta^n(x)\Phi^{m-1}(F(x))(DF(x))^t\nabla\Phi(F(x)).\end{aligned}$$

Plugging this into (2.7) we have

$$\begin{aligned}m\int_G &\langle \operatorname{adj} DF(x)V(F(x))\,,\,(DF(x))^t\nabla\Phi(F(x))\rangle\eta^n(x)\Phi^{m-1}(F(x))\,dx\\ &+ n\int_G \langle (\operatorname{adj} DF(x))(V(F(x)))\,,\,\nabla\eta(x)\rangle\eta^{n-1}(x)\Phi^m(F(x))\,dx\\ &= -\int_G (\operatorname{div} V(F(x)))\eta^n(x)\Phi(F(x))J_F(x)\,dx.\end{aligned}$$

Since $DF(x)\operatorname{adj} DF(x) = J_F(x)I_n$, we have

$$\begin{aligned}\int_G &\langle V(F(x))\,,\,\nabla\Phi(F(x))\rangle\eta^n(x)\Phi^{m-1}(F(x))J_F(x)\,dx\\ &= \frac{n}{-m}\int_G \langle (\operatorname{adj} DF(x))(V(F(x)))\,,\,\nabla\eta(x)\rangle\eta^{n-1}(x)\Phi^m(F(x))\,dx\\ &\qquad + \int_G \frac{\operatorname{div} V(F(x))}{-m}\eta^n(x)\Phi^m(F(x))J_F(x)\,dx.\end{aligned}$$

Suppose that we take a vector field V such that $\operatorname{div} V \leq 0$, after taking absolute values in both sides of the above inequality, we can drop the second term on the right hand side, obtaining

$$\begin{aligned}\int_G &\langle V(F(x))\,,\,\nabla\Phi(F(x))\rangle\eta^n(x)\Phi^{m-1}(F(x))J_F(x)\,dx\\ &\leq \frac{n}{|m|}\int_G \langle (\operatorname{adj} DF)(V(F(x)))\,,\,\nabla\eta(x)\rangle\eta^{n-1}(x)\Phi^m(F(x))\,dx.\end{aligned} \tag{2.8}$$

Suggested by (2.4) we look for smooth vector fields V of class C^1 of the form $|\nabla\Phi(y)|^{n-2}\nabla\Phi(y)$ where Φ is an n-superharmonic function of class $C^2(F(G))$ and $\Phi \geq \delta > 0$. In particular Φ satisfies the inequality

$$\operatorname{div}(|\nabla\Phi(y)|^{n-2}\nabla\Phi(y)) \leq 0.$$

We will show the existence of these functions in the next section.
Substituting V by $|\nabla\Phi|^{n-2}\nabla\Phi$ in (2.8) we have

$$\int_G |\nabla\Phi(F(x))|^n \eta^n(x)\Phi^{m-1}(F(x))J_F(x)\,dx$$
$$\leq \frac{n}{|m|}\int_G |\operatorname{adj} DF(x)|\,|\nabla\Phi(F(x))|^{n-1}|\nabla\eta(x)|\eta^{n-1}(x)\Phi^m(F(x))\,dx.$$

Taking now $m=(1-n)$, the above inequality becomes

$$\int_G \frac{|\nabla\Phi(F(x))|^n}{\Phi^n(F(x))}\eta^n(x)J_F(x)\,dx$$
$$\leq \frac{n}{|m|}\int_G |\operatorname{adj} DF(x)|\frac{|\nabla\Phi(F(x))|^{n-1}}{\Phi^{n-1}(F(x))}|\nabla\eta(x)|\eta^{n-1}(x)\,dx.$$

Using that

$$|\operatorname{adj} DF(x)|^{n/(n-1)} \leq C_n|DF(x)|^n = C_nK(x)J_F(x),$$

for some constant C_n depending only on n. We follow the usual convention of denoting by C_n to be a constant depending only on n which might differ from formula to formula. After applying Hölder's inequality we obtain

$$\int_G \frac{|\nabla\Phi(F(x))|^n}{\Phi^n(F(x))}\eta^n(x)J_F(x)\,dx \leq C_n\int_G |\nabla\eta(x)|^nK^{n-1}(x)\,dx.$$

Since

$$J_F(x)=\frac{|DF(x)|^n}{K(x)}$$

and

$$|\nabla(\Phi\circ F)(x)| \leq |DF(x)|\,|\nabla\Phi(F(x))|$$

we obtain

$$\int_G \frac{|\nabla(\Phi\circ F)(x))|^n}{|(\Phi\circ F)(x)|^n}\eta^n(x)\frac{dx}{K(x)} \leq C_n\int_G |\nabla\eta(x)|^nK^{n-1}(x)\,dx.$$

Hence, we have the following basic estimate:

Theorem 2. *Let $\Phi\in C^2(\Omega')$ such that $\Phi\geq\delta>0$, Φ n-superharmonic with $|\nabla\Phi(y)|^{n-2}\nabla\Phi(y)\in C^1(\Omega')$ and bounded derivative. Then, we have the estimate*

$$\int_G |\nabla(\log\Phi\circ F)(x)|^n\eta^n(x)\frac{dx}{K(x)} \leq C_n\int_G |\nabla\eta(x)|^nK^{n-1}(x)\,dx \tag{2.9}$$

for every $\eta\in C_0^\infty(G)$, where G is a relatively compact open subset of Ω' and $\eta\geq 0$.

3. A smooth n-superharmonic bump.

In this section we give an explicit construction of a family of functions Φ_a that approximates $\log(1/|x|)$ as a approaches 0 in a sense to be made explicit below.

Since our argument is local in nature, without loss of generality we can assume that $F(\Omega) \subset B(0, e^{-e}) = \Omega'$.

Theorem 3. *For each $0 < a < e^{-e}$, there exists a function $\Phi_a : \Omega' \to \mathbb{R}$ with the following properties:*

(i) $\Phi_a \in C^2(\Omega')$,
(ii) $\Phi_a(y) \geq e$, *for every* $y \in \Omega'$,
(iii) Φ_a *is radial,*
(iv) $\Phi'_a(r) = \Phi'_a(|y|) \leq 0$,
(v) Φ_a *is n-superharmonic,*
(vi) $\log(1/a) \leq \Phi_a(y) \leq \log(1/a) + \frac{1}{2} + \log 2$, *for every* $|y| \leq e^{-e}$,
(vii) $\Phi_a(y) = \log(1/|y|)$ *for* $a \leq |y| < e^{-e}$, *and*
(viii) $|\nabla \Phi_a(y)|^{n-2} \nabla \Phi_a(y) \in C^1(\Omega')$.

Proof. Define the functions $\Phi_a(y)$ as follows:

$$\Phi_a(y) = \begin{cases} \log \dfrac{1}{|y|}, & \text{if } r = |y| > a \\ \log \dfrac{1}{a} - \dfrac{|y| - a}{a} + \dfrac{(|y| - a)^2}{2a^2}, & \text{if } \dfrac{a}{2} < |y| < a \\ \log \dfrac{1}{a} + \log 2 + \dfrac{1}{2} + (5 - 12\log 2)\dfrac{|y|^2}{a^2} & \\ + 4(-7 + 12\log 2)\dfrac{|y|^4}{a^4} + 8(5 - 8\log 2)\dfrac{|y|^6}{a^6}, & \text{if } |y| < \dfrac{a}{2}. \end{cases}$$

We arrived at these functions by experimenting at a computer. A *Mathematica* notebook with these calculations is available at

`http://www.pitt.edu/~manfredi/bamsapp.html.`

In the next few lines we provide a classical proof of the proposition. From the definition it is clear that these functions Φ_a are radial. Set

$$u(r) = \log \frac{1}{r} \quad \text{if } r > a,$$

$$v(r) = \log \frac{1}{a} - \frac{r - a}{a} + \frac{(r - a)^2}{2a^2} \quad \text{if } \frac{a}{2} < r < a,$$

$$w(r) = \log \frac{1}{a} + \log 2 + \frac{1}{2} + (5 - 12\log 2)\frac{r^2}{a^2} + 4(-7 + 12\log 2)\frac{r^4}{a^4} + 8(5 - 8\log 2)\frac{r^6}{a^6} \quad \text{if } r < \frac{a}{2}.$$

It is a straightforward exercise to show that the functions u and v agree at a up to order 2, and that v and w agree at $a/2$ up to order 2.

Step 1. The function $v(r)$ is decreasing, indeed $v'(r) < -1/a$ for any $a/2 \leq r \leq a$ as it follows from elementary calculations.

Step 2. The n-Laplacian of v is less than or equal to zero in the interval $(a/2, a)$, i.e. $\Delta_n v \leq 0$.

Proof. Indeed, $|v'(r)| = 1/a + (a-r)/a^2 = (2a-r)/a^2$. Therefore,

$$\begin{aligned}\frac{d}{dr}(r^{n-1}|v'(r)|^{n-2}v'(r)) &= -\frac{d}{dr}(r^{n-1}(-v'(r))^{n-1}) \\ &= -r^{n-2}(n-1)(-v'(r))^{n-2}[-rv''(r) - v'(r)] \\ &= r^{n-2}(n-1)(-v'(r))^{n-2}\left[\frac{r}{a^2} - \frac{1}{a} + \frac{r-a}{a^2}\right] \\ &= r^{n-2}(n-1)(-v'(r))^{n-2}\left[\frac{2(r-a)}{a^2}\right] \leq 0. \qquad \square\end{aligned}$$

Step 3. The function w satisfies $w'(r) < 0$ in the interval $(0, a/2]$.

Proof. Indeed one has

$$aw'(r) = 2(5 - 12\log 2)\left(\frac{r}{a}\right) + 16(-7 + 12\log 2)\left(\frac{r}{a}\right)^3 + 48(5 - 8\log 2)\left(\frac{r}{a}\right)^5.$$

If we let $\frac{r}{a} = s$, then $0 < s \leq \frac{1}{2}$ and set

$$f(s) = aw'(as) = 2(5 - 12\log 2)s + 16(-7 + 12\log 2)s^3 + 48(5 - 8\log 2)s^5.$$

Letting $h(s) = f(s)/s$ we have

$$h(s) = 2(5 - 12\log 2) + 16(-7 + 12\log 2)s^2 + 48(5 - 8\log 2)s^4,$$

where $0 < s \leq \frac{1}{2}$. Let $s^2 = t$, then $0 < t \leq \frac{1}{4}$ and we obtain

$$h(\sqrt{t}) = 2(5 - 12\log 2) + 16(-7 + 12\log 2)t + 48(5 - 8\log 2)t^2.$$

Taking the derivative with respect to t, making it equal to 0 and solving for t, we obtain

$$\frac{d}{dt}h(\sqrt{t}) = 16(-7 + 12\log 2) + 96(5 - 8\log 2)t = 0.$$

This identity holds for $t^* \approx .402855$. The derivative $\frac{d}{dt}h(\sqrt{t})$ at $t = 0$ is approximately equal to 21.08, thus $\frac{d}{dt}h(\sqrt{t}) > 0$ in the interval $(0, \frac{1}{4}]$, and hence the function $h(\sqrt{t})$ is increasing on that interval. If we compute, $h(\sqrt{1/4}) = -3 < 0$ and therefore $w'(r) < 0$ in the interval $(0, a/2]$. $\square$

Step 4. The n-Laplacian of w is less than or equal to zero in the interval $(0, a/2)$, i.e. $\Delta_n w \leq 0$.

Proof. The n-Laplacian for radial functions is given by

$$\Delta_n w = (n-1)r^{-1}(-w'(r))^{n-2}[rw''(r) + w'(r)].$$

Computing the expression in brackets we get,

$$\begin{aligned} &[rw''(r) + w'(r)] \\ &= 2(5 - 12\log 2)\frac{r}{a^2} + 48(-7 + 12\log 2)\frac{r^3}{a^4} + 240(5 - 8\log 2)\frac{r^5}{a^6} \\ &\quad + 2(5 - 12\log 2)\frac{r}{a^2} + 16(-7 + 12\log 2)\frac{r^3}{a^4} + 48(5 - 8\log 2)\frac{r^5}{a^6} \\ &= 4(5 - 12\log 2)\frac{r}{a^2} + 64(-7 + 12\log 2)\frac{r^3}{a^4} + 288(5 - 8\log 2)\frac{r^5}{a^6}. \end{aligned}$$

Therefore, multiplying by a/r and letting $t = r/a$, $0 < t < \frac{1}{2}$, we have

$$h(t) = \frac{a}{t}(taw''(at) + w'(at))$$

and making $s = t^2$, $0 < s < \frac{1}{4}$, we obtain

$$h(\sqrt{s}) = 4(5 - 12\log 2) + 64(-7 + 12\log 2)s + 288(5 - 8\log 2)s^2.$$

Taking the derivative we have

$$\frac{d}{ds}(h(\sqrt{s})) = 64(-7 + 12\log 2) + 576(5 - 8\log 2)s,$$

making it equal to zero and solving for s we obtain

$$s^* = \frac{64(7 - 12\log 2)}{576(5 - 8\log 2)} = \frac{(7 - 12\log 2)}{9(5 - 8\log 2)} = .2685 > \frac{1}{4}.$$

Since $\frac{d}{ds}(h(\sqrt{s}))|_{s=0} = 64(-7 + 12\log 2) = 84.33 > 0$ implies that $\frac{d}{ds}(h(\sqrt{s})) \geq 0$ on the interval $[0, \frac{1}{4}]$ and therefore, the function $h(\sqrt{s})$ is increasing on that interval since $h(\sqrt{1/4}) = -2 < 0$. Thus, we have that $\Delta_n w \leq 0$ on the interval $(0, a/2)$. □

Step 5. The vector field $|\nabla\Phi_a(y)|^{n-2}\nabla\Phi_a(y) \in C^1(\Omega')$ if $n \geq 3$.

Proof. The point $y = 0$ is the only possible problem. There we have,

$$w(r) = a + br^2 + cr^4 + dr^6$$

thus, $w'(r) = 2br + 4cr^3 + 6dr^5$. Therefore,

$$\nabla\Phi_a(y) = \Phi_a'(r)\nabla r = \Phi_a'(r)\frac{y}{r},$$

and hence

$$|\nabla\Phi_a(y)|^{n-2}\nabla\Phi_a(y) = |\Phi_a'(r)|^{n-2}\Phi_a'(r)\frac{y}{r}.$$

Letting

$$\begin{aligned}
V(y) &= |\nabla\Phi_a(y)|^{n-2}\nabla\Phi_a(y) \\
&= |\Phi_a'(r)|^{n-2}\Phi_a'(r)\frac{y}{r} \\
&= (-r(2br + 4cr^3 + 6dr^5))^{n-2}(2br + 4cr^3 + 6dr^5)y \\
&= -r^{n-2}(-(2br + 4cr^3 + 6dr^5))^{n-1}y,
\end{aligned}$$

which is clearly C^∞ away from the origin. We need to show that the partial derivatives of V are in C^1 and bounded at the origin. For that, we compute the partial derivatives

$$\begin{aligned}
\frac{\partial}{\partial x_j}V^i = \frac{\partial}{\partial x_j}(r^{n-2}y^i) &= r^{n-2}\delta_{ij} + (n-2)r^{n-3}\frac{y^i y^j}{r} \\
&= r^{n-2}\delta_{ij} + (n-2)r^{n-4}y^i y^j,
\end{aligned}$$

where δ_{ij} denotes the Kronecker delta function. The first term above is continuous, as for the second term it tends to 0 as $r \to 0$. □

Remark. Theorem 3 is certainly true for $n = 2$ and easier to prove.

4. Completion of the proof of Theorem 1.

Substituting Φ by Φ_a in (2.9) we obtain

$$\int_G |\nabla(\log(\Phi_a \circ F))(x)|^n \eta^n(x)\frac{dx}{K(x)} \leq C_n \int_G |\nabla\eta(x)|^n K^{n-1}(x)dx, \tag{4.1}$$

whenever $\eta \geq 0$, $\eta \in C_0^\infty(G)$. For any $0 \leq \varepsilon < 1$, using the above inequality and Hölder's inequality we have

$$\begin{aligned}
&\int_G |\nabla(\log(\Phi_a \circ F))(x)|^{n-1+\varepsilon}\eta^{n-1+\varepsilon}(x)\,dx \\
&\quad = \int_G |\nabla(\log(\Phi_a \circ F))(x)|^{n-1+\varepsilon}\eta^{n-1+\varepsilon}(x)K(x)\frac{dx}{K(x)} \\
&\quad \leq \left(\int_G |\nabla(\log(\Phi_a \circ F))(x)|^n\eta^n(x)\frac{dx}{K(x)}\right)^{(n-1+\varepsilon)/n} \\
&\qquad\qquad \cdot \left(\int_G K^{(n-1+\varepsilon)/(1-\varepsilon)}(x)\,dx\right)^{(1-\varepsilon)/n}.
\end{aligned}$$

Using (4.1) we obtain

$$\int_G |\nabla(\log(\Phi_a \circ F))(x)|^{n-1+\varepsilon}\eta^{n-1+\varepsilon}(x)\,dx \le C_n \left(\int_G |\nabla\eta(x)|^n K^{n-1}(x)\,dx\right)^{(n-1+\varepsilon)/n} \cdot \left(\int_G K^{(n-1+\varepsilon)/(1-\varepsilon)}(x)\,dx\right)^{(1-\varepsilon)/n}. \tag{4.2}$$

Observe that

$$\frac{n-1+\varepsilon}{1-\varepsilon} = n-1+\frac{n\varepsilon}{1-\varepsilon}.$$

So that if $K(x) \in L^{n-1+\delta}_{\mathrm{loc}}$, we can always find a positive ε so that (4.2) holds. It is worth pointing out that (4.2) always holds for $\varepsilon = 0$.

Without loss of generality we can assume that the function F is nonconstant in the domain Ω, and that $0 \in F(\Omega)$. Thus, there exists $\bar{x} \in \Omega$ such that $|F(\bar{x})| = 2b > 0$. Set $\Omega_b = F^{-1}(|y| < b)$, then Ω_b is open and contains $F^{-1}\{0\}$. Let U be a component of Ω_b such that $U \cap F^{-1}\{0\} \neq \emptyset$. We want to show that $\mathrm{cap}_q(U \cap F^{-1}\{0\}) = 0$, for some $n-1 < q \le n$.

Since $\partial U \subset \partial\Omega$ implies that $U = \Omega$, which is impossible since $\bar{x} \notin U$, there exists $x_0 \in \partial U \cap \Omega \subset \partial\Omega_b \cap \Omega$, hence $|F(x_0)| = b > 0$. Moreover, by continuity $|F(x)| > b/2$ for every $x \in B(x_0,\eta)$ for small $\eta > 0$.

In order to show that $\mathrm{cap}_q(U \cap F^{-1}\{0\}) = 0$ for some $n-1 < q \le n$ it is enough to show that any compact subset of $U \cap F^{-1}\{0\}$ has q-capacity 0, so let K be any compact subset of $U \cap F^{-1}\{0\}$. Choose a small ball B compactly contained in $B(x_0,\eta) \cap U$. We now have that for any $x \in B$

$$e < \log\frac{1}{|F(x)|} < \log\frac{2}{b} < \infty. \tag{4.3}$$

On the other hand by property (vi) of the functions Φ_a's

$$\log\Phi_a(F(x)) = \log\Phi_a(0) \ge \log\frac{1}{a} \tag{4.4}$$

for any $x \in K$.

Select a test function $\eta \in C_0^\infty(U)$ such that $\eta \ge 1$ in K. We consider the function

$$V_a(x) = \frac{\eta(x)\log\Phi_a(F(x))}{\log\frac{1}{a}}.$$

Let us check that $V_a \in W_0^{1,q}(U)$ where $q = n-1+\varepsilon$. We compute the gradient of the function V_a,

$$\nabla V_a(x) = \frac{1}{\log\frac{1}{a}}\{|\nabla(\log\Phi_a \circ F)(x)\eta(x) + (\log\Phi_a \circ F)(x)\nabla\eta(x)\},$$

using equation (4.2) we obtain

$$\begin{aligned}(4.5)\quad \int_U |\nabla V_a(x)|^q dx &\\ \le C_n \frac{1}{(\log \frac{1}{a})^q} \Bigg\{ \left(\int_U |\nabla \eta(x)|^n K^{n-1}(x)\, dx \right)^{q/n} \left(\int_U K^{q/(n-q)}(x)\, dx \right)^{(n-q)/n} &\\ + \int_U |\log \Phi_a \circ F(x)|^q \cdot |\nabla \eta(x)|^q dx \Bigg\}.\end{aligned}$$

The first term on the right hand side of (4.5) is bounded independently of a. In order to estimate the second term observe that $\nabla(\log(\Phi_a \circ F)) \in L^q_{\text{loc}}(\Omega)$, with bounds independent of a. More explicitly we have the following lemma,

Lemma 4. *Whenever $B_r = \mathbb{B}(x,r) \subset B_{2r} = \mathbb{B}(x,2r) \subset \Omega$ and $n-1 < q \le n$ we have*

$$\left(\fint_{B_r} |\nabla(\log(\Phi_a \circ F))(x)|^q dx \right)^{1/q} \le \frac{C_n}{r} \left(\fint_{B_{2r}} K(x)^{q/(n-q)} dx \right)^{(n-q)/q}.$$

Proof. By (4.2) applied to the ball B_{2r} we have

$$\begin{aligned}\int_{B_{2r}} |\nabla(\log(\Phi_a \circ F))(x)|^q \eta^q(x)\, dx \le C_n \left(\int_{B_{2r}} |\nabla \eta(x)|^n K^{n-1}(x) dx \right)^{q/n} &\\ \cdot \left(\int_{B_{2r}} K^{q/(n-q)}(x)\, dx \right)^{(n-q)/n}\end{aligned}$$

where the function $\eta(x)$ is chosen to be 1 on B_r supported in B_{2r} and its gradient behaves as $1/r$. Applying Hölder's inequality to the first integral on the right hand side for $p = q/(n-1)(n-q)$ we obtain

$$\int_{B_r} |\nabla(\log(\Phi_a \circ F))(x)|^q dx \le \frac{C_n}{r^q} \left(\int_{B_{2r}} K^{q/(n-q)}(x)\, dx \right)^{n-q} r^{n(q+1-n)}$$

finally taking the q-th power on both sides and integral averages we have

$$\left(\fint_{B_r} |\nabla(\log(\Phi_a \circ F))(x)|^q dx \right)^{1/q} \le \frac{C_n}{r} \left(\fint_{B_{2r}} K^{q/(n-q)}(x)\, dx \right)^{(n-q)/n} \qquad \square$$

By property (vii) of Theorem 3 we have $|\log(\Phi_a \circ F)(x)| < \log\log(2/b)$, if $a < b/2$ and $x \in B$. It remains to be seen that

$$\int_{\text{supp}\eta} |\log(\Phi_a \circ F)(x)|^q dx$$

is bounded independently of a, for $a < b/2$.

This follows from the following known Sobolev type inequality. We provide an easy proof by the usual method of compactness as in [E].

Lemma 5. *Let G be a domain and $\mathbb{B}$ an open ball contained in G. Then, there exists a constant $C = C(n,p,\mathbb{B},G)$ such that for all $u \in W^{1,p}(G)$ we have*

$$\int_G |u(x)|^p dx \leq C\left\{ \int_G |\nabla u(x)|^p dx + \int_{\mathbb{B}} |u(x)|^p dx \right\}.$$

Proof. By homogeneity, we can assume that $\int_G |u(x)|^p dx = 1$. We need to prove that

$$\int_G |\nabla u(x)|^p dx + \int_{\mathbb{B}} |u(x)|^p dx \geq \frac{1}{C} > 0.$$

Suppose that this is not the case. For every positive integer n, we can find a function $u_n \in W^{1,p}(G)$ satisfying $\int_G |u_n(x)|^p dx = 1$ and

$$\int_G |\nabla u_n(x)|^p dx + \int_{\mathbb{B}} |u_n(x)|^p dx \leq \frac{1}{n}. \tag{4.6}$$

We can now select a subsequence denoted again by u_n and a function $u_0 \in W^{1,p}(G)$ such that u_n converges weakly to u_0 in $W^{1,p}(G)$. It follows that u_n converges to u_0 in $L^p(G)$ and by the weak lower semicontinuity of the L^p norm we have

$$\int_G |\nabla u_0(x)|^p dx \leq \liminf_{n\to\infty} \int_G |\nabla u_n(x)|^p dx = 0.$$

Thus, the function u_0 satisfies $\int_G |u_0(x)|^p dx = 1$ and $\nabla u_0(x) = 0$ for a. e. $x \in G$. Since G is a domain u_0 must be a constant function. On the other hand it follows from inequality (4.6) that $\int_{\mathbb{B}} |u_0(x)|^p dx = 0$ forcing this constant to be 0 in contradiction with $\int_G |u_0(x)|^p dx = 1$. □

Letting $a \to 0$ in (4.5) we conclude that $\operatorname{cap}_q(K) = 0$, and therefore

$$\operatorname{cap}_q(F^{-1}(0)) = 0.$$

By the relationship between Hausdorff dimension and capacity it follows that the Hausdorff dimension of $F^{-1}\{0\}$ is smaller that $n - q < 1$. In particular $F^{-1}\{0\}$ can not contain any segment, and therefore it is totally disconnected. Replacing $F(x)$ by $F(x) - b$ it follows that $F^{-1}\{b\}$ is totally disconnected for any $b \in \mathbb{R}^n$. The mapping F is therefore an orientation preserving light mapping and it follows from the Titus-Young theorem, see [TY], that F is open and discrete.

5. Notes on the critical case.

As we pointed out earlier estimate (4.2) remains valid when we assure only that $K(x) \in L^{n-1}_{\mathrm{loc}}$. We obtain,

$$\int_G \left|\nabla\big(\log(\Phi_a \circ F)\big)(x)\right|^{n-1} \eta^{n-1}(x)\,dx \le C_n \left(\int_G |\nabla\eta(x)|^n K^{n-1}(x)\,dx\right)^{(n-1)/n} \cdot \left(\int_G K^{n-1}(x)\,dx\right)^{1/n}.$$

From this formula, by letting $a \to 0$, it follows that

$$\int_G \left|\nabla\left(\log\log\frac{1}{|F|}\right)(x)\right|^{n-1} dx < \infty.$$

A simple modification of the argument in the previous section gives that the Hausdorff dimension of $F^{-1}(0)$ is less than or equal to 1. With no further restrictions on F we are not able to prove or disprove by means of a counterexample that $F^{-1}(0)$ is a discrete set.

In a related work J. M. Ball, see Theorem 1 and 2 in [B2], showed that if we assume, in addition to $K(x) \in L^p_{\mathrm{loc}}$ for some $p > n-1$, that F has homeomorphic boundary values one can prove that in fact the mapping F is discrete and open. In the same work, Ball constructed some mappings F with homepomorphic boundary values for which $K(x) \in L^p_{\mathrm{loc}}$ for $p < n-1$ which fail to be discrete, leaving open the case $p = n-1$. We conjecture that in this case the mappings are discrete and open.

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