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An Extremal Length Characterization of Closed Sets with Zero Logarithmic Capacity on Quasicircles

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In this note we extend Pflüger's theorem relating logarithmic capacity and extremal length for closed subsets on the unit disc to closed subsets on quasicircles.

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1. INTRODUCTION

We are going to start with some preliminary definitions.

DEFINITION 1 *Let E be a compact set in the complex plane, and let Ω be its complement with boundary $\partial\Omega$. Let $g(z)$ be the Green's function of Ω with pole at ∞. The Green's function $g(z)$ is harmonic in Ω, it vanishes on $\partial\Omega$, and its asymptotic behavior at ∞ is of the form*

$$g(z) = \log|z| + \gamma + \epsilon(z),$$

where γ is a constant and $\epsilon(z) \to 0$ for $z \to \infty$. The constant $\gamma = \gamma(E)$ is called the Robin constant of E.

DEFINITION 2 *Let E_1 and E_2 be two disjoint subsets of an open set Ω. Let Γ be the set of all the rectifiable curves joining E_1 and E_2. Let $P(L)$ be the set of all nonnegative Borel measurable functions such that for any $\rho \in P(L)$,*

$$\int_\lambda \rho(z)|dz| \geq 1,$$

for any $\lambda \in \Gamma$. We define the extremal distance $d_\Omega(E_1, E_2)$ between E_1 and E_2 as

$$d_\Omega(E_1, E_2) = \sup_{\rho \in P(L)} \frac{1}{\iint_\Omega \rho^2(z)\,dx\,dy}.$$

In this note we extend Pflüger's theorem [3] relating the logarithmic capacity of a closed set E on the unit circle $\partial\Delta$ and the extremal distance in Δ between E and a simple closed Jordan curve λ_0 enclosing the origin and contained in the disc $\Delta_{r_0} = \{z \in \Delta : |z| < r_0\}$ where $0 < r_0 < 1$.

THEOREM A *Let E be a closed subset on the unit circle and let $d_\Delta(E,\lambda_0)$ be the extremal distance in Δ between E and λ_0. Then there exist constants C and C' depending only on λ_0 such that*

$$2\gamma(E)+C\leq 2\pi d_\Delta(E,\gamma_0)\leq 2\gamma(E)+C'.$$

Since $cap(E)=e^{-\gamma(E)}$, an immediate consequence of Theorem A is the following Corollary.

COROLLARY A *A closed set E on the unit circle has zero logarithmic capacity if and only if $d_\Delta(E,C_{r_0})=\infty$, for some value r_0, $0<r_0<1$, where $C_{r_0}=\{z\in\Delta:|z|=r_0\}$.*

Let us remark here that Corollary A can be easily proved using the method of the extremal metric, see Lemma 7 in [2].

Our aim is to prove an analog to Theorem A for more general simply connected domains. In particular we consider simply connected domains Ω whose boundary is a quasicircle, that is, if we denote by $\alpha(w_1,w_2)$ the subarc of $\partial\Omega$ of smaller euclidean diameter connecting w_1 and w_2, then there exists a constant C depending only on $\partial\Omega$ such that

$$\operatorname{diam}[\alpha(w_1,w_2)]\leq C|w_1-w_2|.$$

We shall call these domains quasidiscs. For these domains we prove the following results.

THEOREM 1 *Assume that Ω is a quasidisc. Then there exists a point w_0 in Ω and a simple closed Jordan curve λ_{w_0} enclosing w_0 in Ω such that for any closed $E\subset\partial\Omega$,*

$$C_1\gamma(E)+C_2\leq\pi d_\Omega(E,\lambda_{w_0})\leq C_3\gamma(E)+C_4,$$

where the constants C_1 and C_3 depend only on Ω, and C_2 and C_4 also depend on the curve λ_{w_0}.

As an immediate corollary to this theorem we obtain

COROLLARY 1 *Let Ω be a quasidisc. Then for any closed $E\subset\partial\Omega$, E has logarithmic capacity zero if and only if $d_\Omega(\lambda_{w_0},E)=\infty$ for some $w_0\in\Omega$ and λ_{w_0} as in Theorem 1.*

The following theorem and corollary are the generalizations to quasidiscs of theorem 4.9 in [1].

THEOREM 2 *Under the same hypothesis as in Theorem 1, if $C_r=\{w:|w-w_0|=r\}$, $0<r<\frac{1}{2}\operatorname{dist}(w_0,\partial\Omega)$, then for any finite union of closed arcs E on $\partial\Omega$,*

$$\frac{\gamma(E)}{1+k}\leq\pi\lim_{r\to 0}[d_\Omega(C_r,E)-d_\Omega(C_r,\partial\Omega]\leq\frac{\gamma(E)}{1-k}.$$

for some value $0<k<1$, which depends only on the constant of quasiconformality of $\partial\Omega$.

COROLLARY 2 *Under the same hypothesis as in Theorem 2, if V is the set of subharmonic functions $v(z)$ in Ω such that $v\in[C^1(\Omega)\cap C(\overline{\Omega})]$, $v(z)\leq 0$ on E and $v(w_0)\geq 1$, then*

$$\frac{\pi(1-k)}{\gamma(E)}\leq\min_{v\in V}\{D[v]\}\leq\frac{\pi(1+k)}{\gamma(E)},$$

where $D[v]$ is the Dirichlet integral of $v(z)$ in Ω. For a definition of $D[v]$, see [1, *p*. 32].

In section 2 we will give some preliminaries such as definitions and known results which will be used in the proofs. Finally in section 3 we shall prove all the results stated in the introduction.

2. PRELIMINARIES

DEFINITION 3 *We say that a normalized univalent function, $g(z) = z + b_0 + (b_1/z) + \cdots$, defined in the exterior of the unit disc Δ^c, is in the class $\sum(k)$, $0 \le k \le 1$, if it has a homeomorphic extension to C that is K-quasiconformal in the unit disc, where K is a constant depending only on k.*

For this class of functions we have the following result, Theorem 9.14 in [4, p. 292].

THEOREM B *Let $g(z) = z + b_0 + (b_1/z) + \cdots$ be a normalized univalent function defined in the exterior of the unit disc. Then the following are equivalent*:

(a) *$g(\Delta^c)$ is bounded by a quasicircle.*
(b) *$g(z) \in \sum(k)$ for some $0 < k < 1$.*

We will need the following result.

THEOREM 3 *Let $g(z) \in \sum(k)$ for some $0 \le k \le 1$ and $A \subset \partial\Delta$ compact. Then*

$$[cap(A)]^{1+k} \le cap[g(A)] \le [cap(A)]^{1-k}.$$

Proof Following Pommerenke's proof of Theorem 11.7 in [4, p. 346], we find that

$$\{\Delta_n[g(rA)]\}^{1/(n(n-1))} \le r^{1-k}\{\Delta_n(A)\}^{(1-k)/(n(n-1))} \tag{2.1}$$

for any $r > 1$, where

$$\Delta_n(E) = \max_{z_1,\ldots,z_n \in E} \prod_{\mu=1}^{n} \prod_{\substack{\nu=1 \\ \mu\neq\nu}}^{n} |z_\mu - z_\nu|.$$

Since $g(z)$ extends to a homeomorphism in the complex plane, $g(A)$ is compact. Let $z_1, \ldots, z_n$ be points in A such that $\{g(z_\mu)\}_{\mu=1}^{n}$ are nth order Fekete points of $g(A)$, which are the points that maximize $\Delta_n[g(A)]$. Then for fixed n we have

$$\left[\prod_{\mu=1}^{n} \prod_{\substack{\nu=1 \\ \mu\neq\nu}}^{n} |g(z_\mu) - g(z_\nu)|\right]^{1/(n(n-1))} = \lim_{r\to 1}\left[\prod_{\mu=1}^{n} \prod_{\substack{\nu=1 \\ \mu\neq\nu}}^{n} |g(rz_\mu) - g(rz_\nu)|\right]^{1/(n(n-1))}.$$

Thus using (2.1)

$$\{\Delta_n[g(A)]\}^{1/(n(n-1))} \leq \lim_{r\to 1} r^{1-k}\{\Delta_n(A)\}^{(1-k)/(n(n-1))}$$
$$= \{\Delta_n(A)\}^{(1-k)/(n(n-1))},$$

letting $n \to \infty$ in both sides of the above inequality we obtain the second inequality. The first inequality is obtained in a similar way.

After these preliminaries we are ready to prove our results.

3. PROOFS

Proof of Theorem 1

Claim Any closed subset of a quasidisc is compact.

Proof of the Claim Consider the normalized conformal mapping $g(z) : \Delta^c \mapsto \Omega^c$, the exterior of Ω, with a Laurent series $g(z) = z + b_0 + (b_1/z) + \cdots$. Then since by hypothesis $\partial\Omega$ is a quasiconformal curve, by Theorem B, $g(z) \in \sum(k)$ for some value $0 < k < 1$, that is $g(z)$ has a K-quasiconformal extension to the complex plane. This implies that $g^{-1}(E)$ is closed and bounded since $g^{-1}(E) \subset \partial\Delta$, thus $g^{-1}(E)$ is compact. Since $g(g^{-1}(E)) = E$, this shows that E is also compact, and this proves the claim.

Pflüger's theorem applied to $g^{-1}(E)$ tells us that,

$$\gamma(g^{-1}(E)) + \frac{C}{2} \leq \pi d_\Delta(g^{-1}(E), C_{r_0}) \leq \gamma(g^{-1}(E)) + \frac{C'}{2}. \tag{3.1}$$

Since, $\gamma(g^{-1}(E)) = -\log[cap(g^{-1}(E))]$, by Theorem 3 we have that

$$-(1-k)\log[cap(g^{-1}(E))] \leq -\log[cap(E)] \leq -(1+k)\log[cap(g^{-1}(E))].$$

This is equivalent to

$$(1-k)\gamma(g^{-1}(E)) \leq \gamma(E) \leq (1+k)\gamma(g^{-1}(E)) \tag{3.2}$$

It is well known that since $g(z)$ is a quasiconformal map with constant of quasiconformality K, then

$$\frac{1}{K} d_\Omega(E, g(C_{r_0})) \leq d_\Delta(g^{-1}(E), C_{r_0}) \leq K\, d_\Omega(E, g(C_{r_0})). \tag{3.3}$$

Combining (3.1), (3.2) and (3.3) we obtain that

$$\frac{\gamma(E)}{K(1+k)} + \frac{C}{2K} \leq \pi d_\Omega(E, g(C_{r_0})) \leq \frac{\gamma(E)K}{(1-k)} + \frac{C'K}{2}.$$

Let $w_0 = g(0)$ and $\gamma_{w_0} = g(C_{r_0})$. Then we get our desired result

$$C_1\gamma(E) + C_2 \leq \pi\, d_\Omega(E, \gamma_{w_0}) \leq C_3\gamma(E) + C_4,$$

where

$$C_1 = \frac{1}{K(1+k)} \quad \text{and} \quad C_3 = \frac{K}{(1-k)}$$

are constants which only depend on the constant of quasiconformality of Ω, and $C_2 = C/2K$ and $C_4 = C'K/2$ are constants which depend on the constant of quasiconformality of Ω and the simple and closed curve λ_{w_0}.

The proof of Corollary 1 follows directly from Theorem 1 and hence we omit it.

Proof of Theorem 2 By the conformal invariance of extremal distance we have that

$$d_\Omega(C_r,E) - d_\Omega(C_r,\partial\Omega) = d_\Delta(\Phi^{-1}(C_r),\Phi^{-1}(E)) - d_\Delta(\partial\Delta,\Phi^{-1}(C_r)) \tag{3.4}$$

where $\Phi : \Delta \mapsto \Omega$ is a conformal map from Δ to Ω such that $\Phi(0) = w_0$.

Let $\theta_r = \min\{z : |\Phi(z) - w_0| = r\}$ and $\Theta_r = \max\{z : |\Phi(z) - w_0| = r\}$. Applying the comparison principle for extremal length to the right hand side of (3.4) we obtain the following inequalities,

$$\begin{aligned} d_\Delta(\Phi^{-1}(E),C_{\Theta_r}) - d_\Delta(\partial\Delta,C_{\theta_r}) &\le d_\Delta(\Phi^{-1}(E),\Phi^{-1}(C_r)) - d_\Delta(\Phi^{-1}(C_r),\partial\Delta) \\ &\le d_\Delta(\Phi^{-1}(E),C_{\theta_r}) - d_\Delta(\partial\Delta,C_{\Theta_r}), \end{aligned} \tag{3.5}$$

where $C_{\theta_r} = \{z : |z| = \theta_r\}$ and $C_{\Theta_r} = \{z : |z| = \Theta_r\}$. Adding and subtracting $d_\Delta(\partial\Delta, C_{\theta_r})$ from the right hand side of (3.5), and adding and subtracting $d_\Delta(\partial\Delta, C_{\Theta_r})$ from the left hand side of (3.5), and then using equality (3.4), we obtain

$$\begin{aligned} & d_\Delta(\Phi^{-1}(E),C_{\Theta_r}) - d_\Delta(\partial\Delta,C_{\Theta_r}) + \frac{1}{2\pi}\log\left[\frac{\theta_r}{\Theta_r}\right] \\ &\le d_\Omega(E,C_r) - d_\Omega(\partial\Omega,C_r) \\ &\le d_\Delta(\Phi^{-1}(E),C_{\theta_r}) - d_\Delta(\partial\Delta,C_{\theta_r}) + \frac{1}{2\pi}\log\left[\frac{\Theta_r}{\theta_r}\right]. \end{aligned} \tag{3.6}$$

Letting $r \to 0$, it is clear that $\lim_{r\to 0}(1/2\pi)\log[\theta_r/\Theta_r] = 0$ and both sides of (3.6) tend to $(1/\pi)\gamma(\Phi^{-1}(E))$ as r tends to 0 by theorems 4.9 and 2.4 in [1]. Thus

$$\gamma(\Phi^{-1}(E)) = \pi \lim_{r\to 0}\{d_\Omega(E,C_r) - d_\Omega(\partial\Omega,C_r)\}. \tag{3.7}$$

By (3.2) with Φ replaced by g we obtain the desired result.

Corollary 2 follows directly from Theorem 4.9 in [1] and Theorem 3 in the same way than Theorem 2 does, thus we omit the proof.

A natural question to ask is whether the converse of Theorem 1 is true. Namely, if Ω is a simply connected domain such that there exist a closed Jordan curve λ in Ω and constants C_1, C_2, C_3 and C_4 depending only on Ω and λ, with the property that for any closed set $E \subset \partial\Omega$,

$$C_1\gamma(E) + C_2 \le \pi d_\Omega(E,\lambda) \le C_3\gamma(E) + C_4 \tag{3.8}$$

then is Ω a quasidisc?

If this were true, it would give a new characterization for quasidiscs. Unfortunately this is not true as the following construction shows.

Let $\Omega_\epsilon = \Delta \backslash (-1, -1+\epsilon]$ and f_ϵ be the conformal mapping from Ω_ϵ onto the unit disc Δ such that $f_\epsilon(0) = 0$ and $f_\epsilon'(0) > 0$.

It is clear that Ω_ϵ is not a quasidisc. Our goal is to show that for ϵ small enough the inequalities (3.8) are satisfied.

The theory of prime ends and standard results in boundary correspondence for conformal mappings show that f_ϵ is defined on $\partial\Omega_\epsilon$. Let $E \subset \partial\Omega_\epsilon$, then $f_\epsilon(E) \subset \partial\Delta$, and by the conformal invariance of the extremal distance

$$d_{\Omega_\epsilon}(E, \lambda) = d_\Delta(f_\epsilon(E), f_\epsilon(\lambda)).$$

We can apply Schwarz's lemma to the function f_ϵ^{-1}, to obtain

$$|f_\epsilon^{-1}(w)| \le |w|, \qquad |(f_\epsilon^{-1})'(0)| \le 1.$$

By the comparison principle for extremal distances we have that

$$d_{\Omega_\epsilon}(E, f_\epsilon^{-1}(C_{r_0})) \ge d_{\Omega_\epsilon}(E, C_{r_0}) \ge d_{\Omega_\epsilon}(E, C_{f_\epsilon(r_0)}),$$

for any $0 < r_0 < 1$. Let λ be C_{r_0}, then

$$d_{\Omega_\epsilon}(E, C_{r_0}) = d_\Delta(f_\epsilon(E), f_\epsilon(C_{r_0})),$$

and by Pflüger's theorem

$$d_\Delta(f_\epsilon(E), f_\epsilon(C_{r_0})) \ge C_1\gamma(f_\epsilon(E)) + C_2. \tag{3.9}$$

The function $(f_\epsilon^{-1}(z))/((f_\epsilon^{-1})'(0)) \in S$, the class of normalized univalent functions in the unit disc. Thus by [4, p. 351]

$$\frac{1}{|(f_\epsilon^{-1})'(0)|} cap(E) \ge \frac{1}{16}[cap(f_\epsilon(E))]^2.$$

Exponentiating both sides of the above inequality we obtain

$$-\log[|(f_\epsilon^{-1})'(0)|] - \gamma(E) \ge -\log 16 - 2\gamma(f_\epsilon(E)).$$

Thus,

$$\begin{aligned} \gamma(f_\epsilon(E)) &\ge \tfrac{1}{2}[\gamma(E) + \log|(f_\epsilon^{-1})'(0)| - \log 16] \\ &= \frac{1}{2}\left[\gamma(E) + \log\frac{|(f_\epsilon^{-1})'(0)|}{16}\right]. \end{aligned}$$

This allows us to bound (3.9) from below to obtain

$$d_{\Omega_\epsilon}(E, C_{r_0}) = d_\Delta(f_\epsilon(E), f_\epsilon(C_{r_0})) \ge \frac{C_1}{2}\gamma(E) + \frac{C_1}{2}\log\frac{|(f_\epsilon^{-1})'(0)|}{16} + C_2. \tag{3.10}$$

Since $\lim_{\epsilon\to 0}(f_\epsilon^{-1})'(0) = 1$, for ϵ small enough we have that

$$d_{\Omega_\epsilon}(E, C_{r_0}) \ge \frac{C_1}{2}\gamma(E) + C_2',$$

with C_1, C_2' being universal constants.

To get the other inequality, we observe that $\lim_{\epsilon\to 0} f_\epsilon(z) = z$ uniformly in $\overline{\Delta}$. Without loss of generality we can assume that $E \subset \overline{\Delta}$ is compact.

Assume that $\{z_\mu\}_{\mu=1}^n$, are nth order Fekete points of $E \subset \overline{\Delta}$, then

$$[\Delta_n(E)]^{1/(n(n-1)} = \left[\prod_{\nu=1}^{n}\prod_{\substack{\mu=1\\ \mu\neq\nu}}^{n} |z_\nu - z_\mu|\right]^{1/(n(n-1))}$$

$$= \lim_{\epsilon\to 0}\left\{\prod_{\nu=1}^{n}\prod_{\substack{\mu=1\\ \nu\neq\mu}}^{n} |f_\epsilon(z_\nu) - f_\epsilon(z_\mu)|\right\}^{1/(n(n-1))}$$

$$\leq \lim_{\epsilon\to 0}\left\{[\Delta_n(f_\epsilon(E))]^{1/(n(n-1))}\right\}. \tag{3.11}$$

It is not difficult to show that by the symmetry of the function $f_\epsilon(z)$, $f_\epsilon(E)$ is also compact for any positive ϵ, hence

$$\lim_{n\to\infty}[\Delta_n(f_\epsilon(E))]^{1/(n(n-1))} = cap(f_\epsilon(E)).$$

Thus, for each $\delta > 0$ there exists $n_0(\delta,\epsilon) = n_0$ such that

$$[\Delta_{n_0}(f_\epsilon(E))]^{1/(n_0(n_0-1))} \leq cap(f_\epsilon(E)) + \delta,$$

thus substituting this in (3.11) for $n = n_0(\delta,\epsilon) = n_0$,

$$[\Delta_{n_0}(E)]^{1/(n_0(n_0-1))} \leq \lim_{\epsilon\to 0}[cap(f_\epsilon(E))] + \delta.$$

Since $n_0 = n_0(\delta,\epsilon) \to \infty$ as $\delta \to 0$, we obtain that

$$cap(E) \leq \lim_{\epsilon\to 0}[cap(f_\epsilon(E))].$$

Since the limit in ϵ is uniform we can find a positive constant C_0 independent of ϵ and E such that,

$$cap(E) \leq cap(f_\epsilon(E)) + C_0,$$

for ϵ small enough. Thus,

$$\gamma(E) \geq \gamma(f_\epsilon(E)) + C_0' \qquad (C_0' < 0). \tag{3.12}$$

Fix ϵ small enough so that (3.12) holds for any $E \subset \overline{\Delta}$ compact. We take $E \subset \partial\Omega_\epsilon \subset \overline{\Delta}$, then $f_\epsilon(E) \subset \partial\Delta$ and applying Pflüger's theorem we have that

$$C_3\gamma(f_\epsilon(E)) + C_4 \geq d_\Delta(f_\epsilon(E), f_\epsilon(C_{r_0})) = d_{\Omega_\epsilon}(E, C_{r_0})$$

and by (3.12),

$$C_3\gamma(f_\epsilon(E)) + C_4 \leq C_3\gamma(E) + C_4 - C_3C_0' = C_3\gamma(E) + C_4',$$

with C_4' independent of ϵ and E. Hence,

$$d_{\Omega_\epsilon}(E, C_{r_0}) \leq C_3\gamma(E) + C_4',$$

this together with (3.10) give the desired inequalities for some domain Ω_ϵ, which is not a quasidisc.

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