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Analytic Properties of Monotone Sobolev Functions

ENRIQUE VILLAMOR* and BAO QIN LI†

Department of Mathematics, Florida International University, Miami, FL 33199, USA

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We prove that if $u: \mathbb{R}^n \to \mathbb{R}$ is a monotone function in the weighted Sobolev space $W^{1,p}(\mathbb{B}^n; w)$ with $n-1 < p \leq n$, and w a weight in the Muckenhoupt class A_q for $1 \leq q < p/(n-1)$, then u has to be constant. This constitutes a Liouville type theorem for this class of functions. We will also prove a quasi-uniform continuity result for the same class of functions.

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1. INTRODUCTION

Let us start by recalling the definition of monotone functions (in this paper we consider only continuous monotone functions) and of Muckenhoupt A_p weights.

Definition 1.1 Let $\Omega \subset \mathbb{R}^n$ be an open set. A continuous function $u: \Omega \to \mathbb{R}$ is monotone, in the sense of Lebesgue, if

$$\max_{\overline{D}} u(x) = \max_{\partial D} u(x)$$

*Corresponding author. E-mail: villamor@fiu.edu; †libaoqin@fiu.edu

and

$$\min_{\overline{D}} u(x) = \min_{\partial D} u(x)$$

hold whenever D is a domain with compact closure $\overline{D} \subset \Omega$.

Definition 1.2 Let $q > 1$ and $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. We say that $w \in A_q$, if there exists a constant C such that

$$\sup_B \left(⨍_B w(y)\, dy \right) \left(⨍_B w(y)^{1/(1-q)}\, dy \right)^{q-1} < C$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$.

The Sobolev space $W^{1,p}(\mathbb{B}^n; w)$ is defined in [2, Chapter 1]. It consists of functions $u : \mathbb{B}^n \to \mathbb{R}^n$ that have first distributional derivatives ∇u such that

$$\int_{\mathbb{B}^n} (|u(x)|^p + |\nabla u(x)|^p)\, w(x)\, dx < \infty.$$

The weighted p-capacity we will be using throughout this paper is the relative first order variational (p, w)-capacity [2, Chapter 2].

Our first result, already hinted in [3], establishes that a monotone function in the weighted Sobolev space $W^{1,p}(\mathbb{B}^n; w)$ with $n - 1 < p \leq n$, and w a weight in the Muckenhoupt class A_q for $1 \leq q < p/(n-1)$ is quasi-uniformly continuous in the following sense.

THEOREM 1.3 *Let u be a monotone function in the space $W^{1,p}(\mathbb{B}^n; w)$. Suppose that $n - 1 < p \leq n$ and w is Borel weight in the class A_q for some q in the range $1 \leq q < p/(n-1)$. Then, for any $\epsilon > 0$, there exists an open set U in $\mathbb{R}^n$ satisfying $\mathrm{cap}_{p,w}(\mathrm{U}) < \epsilon$ and the function u is uniformly continuous on $\mathbb{B}^n \setminus U$ as a function*

$$u\colon (\mathbb{B}^n, q) \to (\mathbb{R}, d)$$

where q is the hyperbolic metric in $\mathbb{B}^n$ and d is the euclidean metric in $\mathbb{R}$.

From now on, we will call these functions (p, w)-quasicontinuous.

Remark We do not identify w with the equivalence class of measurable functions which agree with w a.e., but rather work with a fixed representative of w that we assume is a Borel function. The reason for this is that we will need to restrict w to $(n-1)$-dimensional sets to define the weighted (p, w)-modulus relative to a hypersurface.

Our second and central result of this paper, is a Sobolev type of inequality for functions in the weighted Sobolev class $W^{1,p}(\mathbb{R}^n; w)$.

THEOREM 1.4 *Let u be a monotone function in the space $W^{1,p}(\mathbb{R}^n; w)$. Suppose that $n-1 < p \leq n$ and w is Borel weight in the class A_q for some q in the range $1 \leq q < p/(n-1)$. Then for any $y \in \mathbb{B}^n(0, R)$ there exists a constant c independent of R such that the following inequality holds*

$$|u(0) - u(y)|^p \leq c \frac{R^p}{\int_{\mathbb{B}^n(0,R)} w(x)\, dx} \int_{\mathbb{B}^n(0,R)} |\nabla u(x)|^p\, w(x)\, dx.$$

As an immediate corollary to this result by letting $R \to \infty$ we obtain the following Liouville type theorem.

COROLLARY 1.5 *Let u be a monotone function in the space $W^{1,p}(\mathbb{R}^n; w)$. Suppose that $n-1 < p \leq n$ and w is Borel weight in the class A_q for some q in the range $1 < p/(n-1)$. Then if*

$$\lim_{R\to\infty} \frac{R^p}{\int_{\mathbb{B}^n(0,R)} w(x)\, dx} = 0$$

we have that the function u is constant.

Our proofs will be based on the modulus method. The limitations $p > n-1$ and w to be a Borel function in the Muckenhoupt class A_q for some $1 \leq q < p/(n-1)$ appear in a modulus estimate, see Lemma 2.3 in [3], on $(n-1)$-dimensional spheres.

In Section 2 we will give some preliminaries and present the proofs of Theorems 1.4 and 1.3 in this order. We are indebted to the anonymous referee for the careful reading of the original manuscript. We have incorporated in this paper several of his suggestions and observations.

2. PRELIMINARIES AND PROOFS OF THEOREM 1.4 AND THEOREM 1.3

The open ball centered at x_0 with radius r is denoted by $\mathbb{B}^n(x_0, r)$. Its boundary is the $(n-1)$-dimensional sphere $S^{n-1}(x_0, r)$. By a cap of a sphere $S^{n-1}(x_0, r)$ we mean a set $H \cap S^{n-1}(x_0, r)$, where H is an open half space in $\mathbb{R}^n$. The spherical distance between two points in $\bar{\mathbb{R}}^n$ is denoted by $q(x, y)$. For a point $x \in \partial\mathbb{B}^n$ we write $C(x)$ for the Stolz cone at x with a fixed given aperture. There exists a constant $c_n \geq 1$, depending only on the aperture and n, such that if $y \in C(x)$ then

$$|y - x| \leq c_n (1 - |y|). \tag{2.1}$$

By $c(\alpha, \beta, \ldots)$ we denote a constant that depends only on the parameters $\alpha, \beta, \ldots$ and that may change value from line to line.

Let Γ be a family of curves in $\mathbb{R}^n$. Denote by $\mathcal{F}(\Gamma)$ the collection of admissible metrics for Γ. These are nonnegative Borel measurable functions $\rho : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ such that

$$\int_\gamma \rho \, ds \geq 1$$

for each locally rectifiable curve $\gamma \in \Gamma$. For $p \geq 1$ the weighted (p, w)-module of Γ is defined by

$$M_p^w(\Gamma) = \inf_{\rho \in \mathcal{F}(\Gamma)} \int_{\mathbb{R}^n} \rho^p \, w \, dx.$$

If $\mathcal{F}(\Gamma) = \phi$, we set $M_p^w(\Gamma) = \infty$. The same definition applies to families of curves that lie in a $(n-1)$-dimensional submanifold S of $\mathbb{R}^n$, replacing the measure $w\,dx$ by $w\,dS$, where dS is the surface measure in the submanifold (note that nothing prevents w from being identically equal to ∞ on the submanifold). The surface module is denoted by $M_p^{w,S}(\Gamma)$.

After these preliminaries we pass to present the proof of Theorem 1.4.

Proof of Theorem 1.4 Let $\mathbb{B}^n(0, R)$ be the ball centered at 0 and radius R. Select a point $y \in \mathbb{B}^n(0, R)$ with $|y| \leq R/2$. Without loss of generality we can assume that $u(0) < u(y)$ (the case $u(0) > u(y)$ is handled by a symmetric argument). Set

$$A = \{z \in \mathbb{B}^n(0, R) : u(z) \leq u(0)\}$$

and

$$B = \{z \in \mathbb{B}^n(0,R): u(z) \geq u(y)\}.$$

Since u is monotone we know that

$$A \cap S^{n-1}(x,t) \neq \emptyset$$

and

$$B \cap S^{n-1}(x,t) \neq \emptyset$$

for $|y| < t \leq R$. From now on let us denote

$$M_p^{w, S^{n-1}(0,t)}\big(\Delta(A \cap S^{n-1}(0,t), B \cap S^{n-1}(0,t); S^{n-1}(0,t))\big)$$

by $M_p^{w, S^{n-1}(0,t)}$. Applying Lemma 2.3 in [3] with $K = S^{n-1}(0,t)$ we obtain

$$c(n,p,q) \leq \frac{1}{t^{((n-p-1)/(q-1))+(n-1)}} \left(M_p^{w, S^{n-1}(0,t)}\right)^{1/(q-1)} \times \left(\int_{S^{n-1}(0,t)} w(y)^{1/(1-q)}\, dS(y)\right),$$

for any t between $|y| < t \leq R$. Using the fact that $(n-p-1)/(q-1) + (n-1)$ is negative and since w is a positive weight, integrating the above inequality between $|y|$ and R we get

$$\int_{|y|}^{R} \left(\frac{1}{M_p^{w, S^{n-1}(0,t)}}\right)^{1/(q-1)} \leq c(n,p,q)\, R^{((p+1-n)/(q-1))-(n-1)} \times \left(\int_{\mathbb{B}^n(0,R)} w(x)^{1/(1-q)}\, dx\right). \tag{2.2}$$

Using Hölder's inequality, the definition of the modulus, exponentiating the resulting inequality to the power $q/q-1$ and using the monotonicity property of the modulus in (2.2) we obtain that

$$\frac{R^{q/(q-1)}}{\left[M_p^w\big(\Delta(A, B, B^n(0,R)\big)\right]^{1/(q-1)}} \leq c R^{((p+1-n)/(q-1))-(n-1)} \times \left(\int_{\mathbb{B}^n(0,R)} w(x)^{1/(1-q)}\, dx\right).$$

Since by assumption the weight $w \in A_q(\mathbb{R}^n)$, using the A_q-condition we have that

$$\frac{R^{q/(q-1)}}{\left[M_p^w\left(\Delta\left(A, B, B^n(0, R)\right)\right)\right]^{1/(q-1)}} \leq cR^{((p+1-n)/(q-1))-(n-1)}$$
$$\times \frac{R^{nq/(q-1)}}{\left(\int_{\mathbb{B}^n(0,R)} w(x)\,dx\right)^{1/(q-1)}},$$

where c is a constant depending on n, p, q and w. Combining the powers of R we obtain

$$M_p^w\left(\Delta\left(A, B, B^n(0, R)\right)\right) \geq cR^{-p} \int_{\mathbb{B}^n(0,R)} w(x)\,dx. \tag{2.3}$$

By Lemma 2.2 in [3] we obtain

$$M_p^w\left(\Delta\left(A, B, B^n(0, R)\right)\right) \leq \frac{1}{|u(x)-u(y)|^p} \int_{B^n(0,R)} |\nabla u(x)|^p\, w(x)\,dx. \tag{2.4}$$

Combining inequalities (2.3) and (2.4) we obtain

$$R^{-p} \int_{\mathbb{B}^n(0,R)} w(x)\,dx \leq \frac{1}{|u(x)-u(y)|^p} \int_{B^n(0,R)} |\nabla u(x)|^p\, w(x)\,dx,$$

which can be rewritten as

$$|u(x)-u(y)|^p \leq c\,\frac{R^p}{\int_{\mathbb{B}^n(0,R)} w(x)\,dx} \int_{B^n(0,R)} |\nabla u(x)|^p\, w(x)\,dx. \qquad \square$$

Our Corollary 1.5 follows immediately from this inequality. It is worth to observe that in the case our weight w is identically equal to 1 then

$$\lim_{R\to\infty} \frac{R^p}{\int_{\mathbb{B}^n(0,R)} w(x)\,dx} = \lim_{R\to\infty} R^{p-n},$$

which goes to 0 whenever $p < n$. In the case $p = n$ the limit to be considered is

$$\lim_{R \to \infty} \frac{1}{\ln(R)}$$

which also goes to 0 as $R \to \infty$. Thus, Theorem 1.4 implies that the only monotone Sobolev functions in $\mathbb{R}^n$ are the constant functions.

We pass now to prove Theorem 1.3.

Proof of Theorem 1.3 It is well known that if $w \in A_q$ and $q < p$ then $w \in A_p$. By Theorem D in [1] we can extend u to a function f in $W^{1,p}(\mathbb{R}^n; w)$ such that

$$\int_{\mathbb{R}^n} |\nabla f(x)|^p \, w(x)\, dx \le c \int_{B^n(0,1)} |\nabla u(x)| \, w(x)\, dx,$$

for some constant c depending on n, p and the A_p constant of the weight w. We continue to denote this extension by u. Fix $\epsilon > 0$ and choose U as in Lemma 3.3 in [3]. For a constant δ_0 to be determined later, choose $r_0 > 0$ such that for $0 < r < r_0$ and $x_0 \in \overline{\mathbb{B}}^n(0,1) \setminus U$,

$$\int_{\mathbb{B}^n(x_0,r)} |\nabla u(y)|^p \, w(y) dy \le \delta_0 \, r^{-p} \, w(\mathbb{B}^n(x_0,r)). \tag{2.5}$$

We want to show that for any $x \in \mathbb{B}^n \setminus U$ and any positive δ, there exists a positive M such that whenever $y \in \mathbb{B}^n(x, M(1-|x|))$ we have that

$$|u(x) - u(y)| < \delta.$$

For this select an arbitrary point $x_0 \in \mathbb{B}^n \setminus U$. Select now a point $y \in \mathbb{B}^n(x_0, (1-|x_0|)/4)$. We may assume that $u(x_0) < u(y)$ (the case $u(x_0) > u(y)$ is handled by a symmetric argument). Set

$$A = \left\{ z \in \mathbb{B}^n\left(x_0, \frac{1-|x_0|}{2}\right) : u(z) \le u(x_0) \right\}$$

and

$$B = \left\{ z \in \mathbb{B}^n\left(x_0, \frac{1-|x_0|}{2}\right) : u(z) \ge u(y) \right\}.$$

Since u is monotone we know that

$$A \cap S^{n-1}(x_0, t) \neq \emptyset$$

and

$$B \cap S^{n-1}(x_0, t) \neq \emptyset$$

for $|x_0 - y| < t \leq (1 - |x_0|)/2$. As in Theorem 1.3, under these circumstances we obtain the following inequality

$$\begin{aligned} \frac{1}{|u(x_0) - u(y)|^p} \int_{\mathbb{B}^n(x_0,(1-|x_0|)/2)} |\nabla u(x)|^p \, w(x)\, dx \\ \geq \frac{c}{(1 - |x_0|)^p} \int_{\mathbb{B}^n(x_0,(1-|x_0|)/2)} w(y)\, dy. \end{aligned}$$

Since $x_0 \in \overline{\mathbb{B}}^n(0,1) \setminus U$ and choosing $r = (1 - |x_0|)/2 < r_0$ in (2.5) we obtain

$$\begin{aligned} \frac{1}{|u(x_0) - u(y)|^p}\, \delta_0 \,(1 - |x_0|)^{-p}\, w\left(\mathbb{B}^n\left(x_0, \frac{1 - |x_0|}{2}\right)\right) \\ \geq \frac{c}{(1 - |x_0|)^p} \int_{\mathbb{B}^n(x_0,(1-|x_0|)/2)} w(y)\, dy, \end{aligned}$$

where we have denoted $\int_{\mathbb{B}^n(x_0,(1-|x_0|)/2)} w(y)\, dy$ by $w(\mathbb{B}^n(x_0, (1-|x_0|)/2))$. This is equivalent to say that

$$|u(x_0) - u(y)|^p \leq c\, \delta_0 = \tilde{\delta}.$$

Choosing δ such that $c^{1/p}\,\delta_0^{1/p} = \delta$ we obtain our desired result for points $x_0 \in \mathbb{B}^n \setminus U$ near the boundary of $\mathbb{B}^n$, for the remaining points inside $\mathbb{B}^n$ the result is trivial by uniform continuity. □

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