BOUNDARY LIMITS FOR BOUNDED QUASIREGULAR MAPPINGS

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ABSTRACT. In this paper we establish results on the existence of nontangential limits for weighted $\mathcal{A}$-harmonic functions in the weighted Sobolev space $W_w^{1,q}(\mathbb{B}^n)$, for some $q > 1$ and w in the Muckenhoupt A_q class, where $\mathbb{B}^n$ is the unit ball in $\mathbb{R}^n$. These results generalize the ones in section §3 of [KMV], where the weight was identically equal to one. Weighted $\mathcal{A}$-harmonic functions are weak solutions of the partial differential equation

$$\operatorname{div}(\mathcal{A}(x, \nabla u)) = 0,$$

where $\alpha\, w(x)\, |\xi|^q \leq \langle \mathcal{A}(x,\xi), \xi \rangle \leq \beta\, w(x)\, |\xi|^q$ for some fixed $q \in (1, \infty)$, where $0 < \alpha \leq \beta < \infty$, and $w(x)$ is a q-admissible weight as in Chapter 1 in [HKM].

Later, we apply these results to improve on results of Koskela, Manfredi and Villamor [KMV] and Martio and Srebro [MS] on the existence of radial limits for bounded quasiregular mappings in the unit ball of $\mathbb{R}^n$ with some growth restriction on their multiplicity function.

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§1. Introduction.

In this paper we study weak solutions of the partial differential equation

$$\operatorname{div}(\mathcal{A}(x, \nabla u)) = 0, \tag{1.1}$$

where $\mathcal{A}: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a mapping satisfying the following assumptions for some constants $0 < \alpha \leq \beta < \infty$:

$$\text{the mapping } x \to \mathcal{A}(x, \xi) \text{ is measurable for all } \xi \in \mathbb{R}^n \text{ and}$$

$$\text{the mapping } \xi \to \mathcal{A}(x, \xi) \text{ is continuous for a.e. } x \in \mathbb{R}^n; \tag{1.2}$$

for all $\xi \in \mathbb{R}^n$ and a. e. $x \in \mathbb{R}^n$

$$\langle \mathcal{A}(x, \xi), \xi \rangle \geq \alpha\, w(x)\, |\xi|^q \tag{1.3}$$

$$|\mathcal{A}(x, \xi)| \leq \beta\, w(x)\, |\xi|^q \tag{1.4}$$

where $1 < q < \infty$;

$$\langle (\mathcal{A}(x, \xi_1) - \mathcal{A}(x, \xi_2)), (\xi_1 - \xi_2) \rangle > 0 \tag{1.5}$$

whenever ξ_1, $\xi_2 \in \mathbb{R}^n$, $\xi_1 \neq \xi_2$; and

$$\mathcal{A}(x, \lambda \xi) = \lambda\, |\lambda|^{q-2}\, \mathcal{A}(x, \xi) \tag{1.6}$$

whenever $\lambda \in \mathbb{R}$, $\lambda \neq 0$. More generally we could have replaced in (1.3) and (1.4) β by a function $\beta(x)$ with the condition that is bounded and α by a function $\alpha(x)$ asking that $\alpha(x) > 0$ a. e. x. Instead, we will consider the uniformly elliptic case and general $1 < q < \infty$. Here we assume that $w(x)$ is a q-admissible nonnegative weight as defined in Chapter 1 of [HKM].

Solutions of (1.1) are called weighted $\mathcal{A}$-harmonic functions. The prototype of these equations is the weighted p-Laplace equation

$$\Delta_{q,w} u = \operatorname{div}(w(x)\, |\nabla u|^{q-2} \nabla u) = 0.$$

In this note we present generalizations of a number of theorems on the existence of nontangential limits of weak solutions of (1.1) in a ball $\mathbb{B}$ with finite weighted q-Dirichlet integral

$$\int_{\mathbb{B}} |\nabla u|^q\, w(x)\, dx < \infty,$$

for some $q > 1$ and w a nonnegative weight in the Muckenhoupt A_q class.

Notice that for $q > n - 1$ monotonicity of the functions in the weighted Sobolev class is all it is needed as is shown in [MV1]. It is well known that solutions of (1.1) satisfying conditions (1.2) through (1.6) are monotone. Hence, the real contribution of this note for solutions of (1.1) satisfying conditions (1.2) through (1.6) is in the range $1 < q \leq n - 1$.

The weighted Sobolev class $W^{1,q}(\mathbb{B}^n; w)$ is defined in [HKM, Chapter 1]. It consists of functions $u\colon \mathbb{B}^n \to \mathbb{R}^n$ that have first distributional derivatives ∇u such that

$$\int_{\mathbb{B}^n} \left(|u(x)|^q + |\nabla u(x)|^q\right) \, w(x) \, dx < \infty.$$

The weighted q-capacity we will be using throughout this paper is the relative first order variational (q, w)-capacity [HKM, Chapter 2]. Let us recall for the sake of completeness the definitions of monotone functions and of Muckenhoupt A_q weights.

Definition 1.7. *Let $\Omega \subset \mathbb{R}^n$ be an open set. A continuous function $u\colon \Omega \to \mathbb{R}$ is monotone, in the sense of Lebesgue, if*

$$\max_{\overline{D}} u(x) = \max_{\partial D} u(x)$$

and

$$\min_{\overline{D}} u(x) = \min_{\partial D} u(x)$$

hold whenever D is a domain with compact closure $\overline{D} \subset \Omega$.

Definition 1.8. *Let $q > 1$ and $w \in L^1_{\text{loc}}(\mathbb{R}^n)$. We say that $w \in A_q$, if there exists a constant C such that*

$$\sup_B \left(⨍_B w(y)\, dy \right) \left(⨍_B w(y)^{\frac{1}{1-q}}\, dy \right)^{q-1} < C$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$.

Let us observe that if the weight w is in A_q it follows that it is q-admissible with the same index q, see Chapter 15 in [HKM].

Our results extend the ones in section §3 of [KMV], where the weight function w was identically equal to one. We refer to the introduction of [KMV], for a historical chronology, background and references for these type of results.

In section §2 we generalize the results in section §3 of [KMV] for the weighted case.

In section §3 we apply the results in section §2 and in [MV1] to the components of bounded quasiregular mappings f satisfying certain growth conditions on their multiplicity function $N(f,E)$ defined as follows: Let E be a subset of the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$. We define $n(y;f,E) = \operatorname{card}\{x \in E: f(x) = y\}$, and $N(f,E) = \sup_{y\in\mathbb{R}^n} n(y;f,E)$. $N(f,E)$ is called the *multiplicity function* of f.

Our main result appears in this section, and is the following:

Theorem 3.11. *Let f be a bounded quasiregular mapping of $\mathbb{B}^n$, and suppose that, for some $0 \le a < n-1$,*

$$N(f, B(0,r)) \le C\,(1-r)^{-a}$$

for all $0 < r < 1$. Then the set of points $x_0 \in E \subset \partial\mathbb{B}^n(0,1)$ for which the nontangential limit of f does not exist has Hausdorff dimension less than or equal to a, i.e. $dim_H(E) \le a$.

This theorem improves Theorem 4.1 in [KMV] where the bound on the Hausdorff dimension of E was found to be $\frac{n\,a}{1+a}$. It is clear that for any $0 < a < n-1$ we have that $a < \frac{n\,a}{1+a}$.

In 1999, Martio and Srebro [MS] showed that for a bounded quasiregular mapping in the unit ball of $\mathbb{R}^n$ satisfying the same condition as in Theorem 3.11 for its multiplicity function , if f is locally injective, then $\dim_H(E) \le a$, and that this estimate is best possible. For the sharpness of their estimate, they constructed for every $n \ge 3$ a sequence of numbers $s_m \in (0, n-1)$ and locally injective bounded quasiregular mappings f_m in $\mathbb{B}^n$, $m = 1, 2, \ldots$ such that f_m satisfies that

$$N(f_m, B(0,r)) \le C\,(1-r)^{-s_m}$$

and $\dim_H(E_m) = s_m$ with $\lim_{m\to\infty} s_m = n-1$, where E_m is the set of points $x_0 \in E_m \subset \partial\mathbb{B}^n(0,1)$ for which the nontangential limit of f_m does not exist. These examples also show that our result is sharp.

Theorem 3.11 improves on Martio and Srebro's in that we do not assume in ours that the bounded quasiregular mappings are locally injective. As it will be shown at the end of section §3, our result also holds with the same conclusion, for quasiregular mappings with the same restriction on the growth of their multiplicity function but not necessarily bounded, i.e. we can assume that $|f(x)| \le C\,(1-|x|)^{-b}$ for some $0 < b < \infty$ and Theroem 3.11 still holds with the same conclusion. Martio and Srebro's result only holds for locally injective and bounded qausiregular mappings with restricted growth in their multiplicity function. Thus, the natural question to ask, is whether or not Theorem 3.11 holds for $a = n-1$, that is:

Is it true that a bounded quasiregular mapping satisfying that

$$N(f, B(0,r)) \leq C\,(1-r)^{-(n-1)}$$

has nontangential boundary limits everywhere on the boundary of the unit ball except possibly on a set of $(n-1)$ Hausdorff measure zero?

Recently Heinonen and Rickman [HR], have constructed examples in dimension $n = 3$, of bounded quasiregular mappings, not necessarily locally injective, with no radial limits at points in sets of the boundary of the $\mathbb{B}^3$ with Hausdorff dimension arbitrarily close to 2.

§2. Existence of Nontangential Limits

In this section we prove straightforward generalizations of results in section §3 of [KMV] on boundary limits for weighted Dirichlet finite $\mathcal{A}$-harmonic functions. In [KMV] no weight was considered. Throughout this section we assume that $\alpha(x) > \alpha > 0$ for a. e. x and that our weights w are in the A_q Muckenhoupt class.

First we show that weighted Dirichlet finite $\mathcal{A}$-harmonic functions u, defined in the unit ball $\mathbb{B}^n$ of $\mathbb{R}^n$, have nontangential limits everywhere on the boundary of the unit ball except possibly on a set E of weighted Bessel $B^w_{1,q}$-capacity zero, $1 < q \leq n$. In this work we will use the weighted Bessel $B^w_{1,q}$-capacity for technical reasons. We refer the reader to the book by Ziemer [Z] for the definition and properties of the weighted Bessel capacity $B^w_{1,q}$, where the weight $w(x) \in A_q(\mathbb{R}^n)$. At this point, we would like to remark that all the (q, w)-capacities are equivalent in the sense that a set with one of the standard (q, w)-capacities zero will have all the other (q, w)-capacities zero. Thus, in the rest of the paper we will say (q, w)-capacity zero without specifying the weighted capacity that we are using. The case $q > n$ is not interesting because then u is continuous up to the boundary by the weighted Sobolev embedding theorem. Recall that a weighted $\mathcal{A}$-harmonic function u of $\mathbb{B}^n$ is continuous in $\mathbb{B}^n$.

Theorem 2.1. *Let u be a weighted $\mathcal{A}$-harmonic function in the unit ball $\mathbb{B}^n$ of $\mathbb{R}^n$ (no restriction on the type of $\mathcal{A}$). If $\int_{\mathbb{B}^n} |\nabla u(x)|^q\, w(x)\, dx < \infty$ for some $1 < q \leq n$ and $w(x) \in A_q(\mathbb{R}^n)$, then the function u has nontangential limits on all radii terminating outside a set of (q, w)-capacity zero.*

Theorem 2.1 extends Theorem 3.1 in [KMV] where no weight was considered. The proof of Theorem 2.1 is based on the following two lemmas, which are straightforward generalizations of Lemma 3.2 and Lemma 3.4 in [KMV] respectively.

Lemma 2.2. *Let $u \in W^{1,q}_w(\mathbb{R}^n)$, $1 < q \le n$ and $w(x) \in A_q(\mathbb{R}^n)$. Then*

$$\lim_{r\to 0} \fint_{B(x,r)} |u(y) - u(x)|^q \, w(y)\, dy = 0 \tag{2.3}$$

except for x in a set $E \subset \mathbb{R}^n$ of (q,w)-capacity zero.

In this paper we denote $\frac{1}{\int_{B(x,r)} w(y)\,dy} \int_{B(x,r)}$ by $\fint_{B(x,r)}$.

The proof of this lemma is a straightforward adaptation of the proof of Lemma 3.2 in [MV1], replacing the initial definition in the proof of that lemma of $A_r u(x)$ by the new definition

$$A_r u(x) = r^q \fint_{B(x,r)} |u(y) - u(x)|^q \, w(y)\, dy,$$

and using then Lemma 3.1 in [MV1] and the fact that smooth functions with compact support are dense in $W^{1,q}_w(\mathbb{R}^n)$ whenever the weight w is in the class A_q (see [K]).

Lemma 2.4. *[HKM, Theorem 3.34] Let u be a weighted $\mathcal{A}$-harmonic function in $\mathbb{B}^n$, and fix $1 < q \le n$ and $w(x) \in A_q(\mathbb{R}^n)$. Then, there exists a constant C such that for each ball $B = B(x,r) \subset \mathbb{B}^n$ and all $a \in \mathbb{R}$*

$$\sup_{\frac{1}{2}B} |u(y) - a| \le C \, \Big(\fint_B |u(y) - a|^q \, w(y)\, dy\Big)^{1/q},$$

where $\frac{1}{2}B = B(x, r/2)$.

Proof of Theorem 2.1. Since $\int_{\mathbb{B}^n} |\nabla u(x)|^q \, w(x)\, dx < \infty$, it follows from the Poincaré inequality that $u \in W^{1,q}_w(\mathbb{B}^n)$. Hence, by standard extension theorems, we may assume that $u \in W^{1,q}_w(\mathbb{R}^n)$. We show that u has a nontangential limit for each $x \in \partial\mathbb{B}^n$ for which (2.3) holds. The claim then follows from Lemma 2.2.

Fix a $w \in \partial\mathbb{B}^n$ for which (2.3) holds. We denote by $C(w)$ the Stolz cone at w with a fixed given aperture. Then we can find a constant $c_n \ge 1$, depending only on the aperture and n, such that for all $x \in C(w)$

$$|w - x| \le c_n(1 - |x|).$$

Pick $x \in C(w)$. Then, we have that

$$B(x, (1-|x|)/2) \subset B(w, (c_n + \frac{1}{2})(1 - |x|)),$$

and hence, by Lemma 2.4,

$$\begin{aligned}|u(x)-u(w)| &\le C\,\Big(\fint_{B(x,(1-|x|)/2)} |u(y)-u(w)|^q\,w(y)\,dy\Big)^{1/q}\\ &\le C'\,\Big(\fint_{B(w,(c_n+\frac{1}{2})(1-|x|))} |u(y)-u(w)|^q\,w(y)\,dy\Big)^{1/q}.\end{aligned}$$

The claim follows by applying (2.3). □

In Theorem 2.1, $\mathcal{A}$-harmonicity was not essential but merely the version of the weak weighted Harnack inequality (Lemma 2.4) satisfied by the solutions to a large class of elliptic nonlinear P.D.E.'s.

§3. Nontangential limits for Quasiregular Mappings.

Let $W^{1,n}_{\mathrm{loc}}(\mathbb{B}^n)$ denote the local Sobolev space of functions in $L^n_{\mathrm{loc}}(\mathbb{B}^n)$ whose distributional derivatives belong to $L^n_{\mathrm{loc}}(\mathbb{B}^n)$. Consider a mapping

$$f\colon \mathbb{B}^n \to \mathbb{R}^n$$

whose coordinate functions belong to $W^{1,n}_{\mathrm{loc}}(\mathbb{B}^n)$. Denote by $J_f(x)$ the Jacobian determinant $\det(Df(x))$. For a.e. $x \in \mathbb{B}^n$ the *dilatation* of f is defined by

$$K(x) = \frac{|Df(x)|^n}{J_f(x)},$$

and it satisfies $K(x) \ge c_n$. If $K(x) \in L^\infty(\mathbb{B}^n)$, then f is said to be a quasiregular mapping.

It is well known, see [HKM], that if f is a nonconstant K-quasiregular mapping in $\mathbb{B}^n$ and $b \in \mathbb{R}^n$, the function $u(x) = \log|f(x)-b|$ is a weighted $\mathcal{A}$-harmonic function in $\mathbb{B}^n \setminus f^{-1}(b)$ of type $q = n$, weight $w = 1$, $\alpha = \frac{1}{K}$, and $\beta = K$. Therefore, all the results of the previous section apply to quasiregular mappings.

In this section we prove that a certain restriction on the growth of the multiplicity function of f implies the existence of nontangential limits.

After these preliminaries, let us prove the following result.

Theorem 3.1. *Let f be a bounded quasiregular mapping of $\mathbb{B}^n$. Let w be a nonnegative weight defined by*

$$w(x) = \sum_{j=0}^{\infty} c_j\,|1-|x||^{q-1}\,\aleph_{R_j}(x), \tag{3.2}$$

where $1 < q < n$ and $R_j = \{x\colon 1-2^{-j} < |x| \le 1-2^{-j-1}\}$, $j = 0,1,2,\dots$, $\aleph_{R_j}$ is the characteristic function of R_j, and the c_j's are positive constants *satisfying that for some positive* b $\sum_{j=1}^{\infty} \frac{c_j}{j^b \frac{q}{n}} < \infty$. *Let us assume that for the same positive* b *we have that*

$$N(f, B(0,r)) \le C\,(1-r)^{-(n-1)} \left(\frac{1}{\log\left(\frac{1}{1-r}\right)} \right)^b \tag{3.3}$$

for all $0 < r < 1$.

Then

$$\int_{\mathbb{B}^n(0,1)} |Df(x)|^q\, w(x)\, dw < \infty.$$

It is important to observe that here we do not need to assume that the weight $w(x)$ as defined in (3.2) is in $A_q(\mathbb{R}^n)$.

Proof of Theorem 3.1. Let $R_j = \{x: 1 - 2^{-j} < |x| \le 1 - 2^{-j-1}\}$, $j = 1, 2, \ldots$. Now

$$\sum_{j=1}^{\infty} \int_{R_j} |Df(x)|^q\, w(x)\, dx \le C \sum_{j=1}^{\infty} \Big(\int_{R_j} |Df(x)|^n\, dx\Big)^{q/n} \Big(\int_{R_j} w(x)^{\frac{n}{n-q}}\, dx\Big)^{\frac{n-q}{n}},$$

and by a change of variables [BI, 8.3] we arrive at

$$\sum_{j=1}^{\infty} \int_{R_j} |Df(x)|^q\, w(x)\, dx \le C \sum_{j=1}^{\infty} \Big(\int_{f(R_j)} n(y, f, R_j)\, dy\Big)^{q/n} \Big(\int_{R_j} w(x)^{\frac{n}{n-q}}\, dx\Big)^{\frac{n-q}{n}}.$$

Since

$$\begin{aligned} n(y, f, R_j) &\le N(f, B(1 - 2^{-j-1})) \\ &\le C\,(1 - (1 - 2^{-j-1}))^{-(n-1)} \left(\frac{1}{\log\left(\frac{1}{1-(1-2^{-j-1})}\right)} \right)^b \\ &= C\, 2^{j(n-1)} \frac{1}{j^b}, \end{aligned}$$

and since $w(x) = \sum_{j=0}^{\infty} c_j\, |1 - |x||^{q-1} \aleph_{R_j}(x)$, then in R_j,

$$w(x) \le C\, 2^{-j(q-1)}\, c_j.$$

Thus

$$\sum_{j=1}^{\infty} \int_{R_j} |Df(x)|^q\, w(x)\, dx \le C \sum_{j=1}^{\infty} 2^{j(n-1)\frac{q}{n}} \frac{1}{j^{b\frac{q}{n}}} 2^{-j(\frac{n-q}{n})} 2^{-j(q-1)} c_j.$$

Thus, after simplifying we have that

$$\begin{aligned}\sum_{j=1}^{\infty} \int_{R_j} |Df(x)|^q\, w(x)\, dx &\le C \sum_{j=1}^{\infty} \frac{c_j}{j^{b\frac{q}{n}}} \\ &= C \sum_{j=1}^{\infty} \frac{c_j}{j^{b\frac{q}{n}}} < \infty,\end{aligned}$$

by assumption. Therefore $\int_{\mathbb{B}^n \setminus B(0,1/2)} |Df(x)|^q\, w(x)\, dx < \infty$, thus

$$\int_{\mathbb{B}^n} |Df(x)|^q\, w(x)\, dx < \infty$$

and the claim will follow. □

Let w be a weight as in Theorem 3.1 with the constants c_j, $j = 1, 2, \ldots$ satisfying the hypothesis of the theorem. Let us assume at this stage that w belongs to $A_q(\mathbb{R}^n)$.

It follows from [A, Theorem 6.1] that

$$(3.4) \qquad (q, w) - \operatorname{cap}(\mathbb{B}^n(x, r)) \approx \left[\int_r^{\infty} t^{\frac{(q-n)}{(q-1)}} \fint_{\mathbb{B}(x,t)} w^{-\frac{1}{q-1}} \frac{dt}{t} \right]^{1-q}.$$

Since we are assuming that $w \in A_q$, we have that

$$\fint_{\mathbb{B}(x,t)} w^{-\frac{1}{q-1}} \le C \frac{1}{\left(\fint_{\mathbb{B}(x,t)} w \right)^{\frac{1}{q-1}}},$$

and thus dividing by something bigger,

$$(3.5) \qquad (q, w) - \operatorname{cap}(\mathbb{B}^n(x, r)) \ge C \left[\int_r^{\infty} t^{\frac{(q-n)}{(q-1)}} \frac{1}{\left(\fint_{\mathbb{B}(x,t)} w \right)^{\frac{1}{q-1}}} \frac{dt}{t} \right]^{1-q}.$$

Using now the explicit formula for our weight w, we have that

$$w(\mathbb{B}(x,t)) \approx \sum_{j=j_0}^{\infty} 2^{-j(n+q-1)}\, c_j \approx t^{n+q-1}\, c_{j_0(t)},$$

where $j_0(t)$ is a positive integer satisfying that $t \approx 2^{-j_0(t)}$. By definition

$$\fint_{\mathbb{B}(x,t)} w = \frac{1}{t^n}\, w(\mathbb{B}(x,t)),$$

then, substituting in (3.5) we have that

$$(q,w) - \ \mathrm{cap}(\mathbb{B}^n(x,r)) \geq C \left[\int_r^{\infty} t^{\frac{(q-n)}{(q-1)}} \frac{1}{\left(t^{q-1}\, c_{j_0(t)}\right)^{\frac{1}{q-1}}} \frac{dt}{t} \right]^{1-q}.$$

Since for any t between r and ∞ we can assume without loss of generality that

$$\frac{1}{\left(t^{q-1}\, c_{j_0(t)}\right)^{\frac{1}{q-1}}} \leq \frac{1}{\left(r^{q-1}\, c_{j_0(r)}\right)^{\frac{1}{q-1}}},$$

thus

$$(q,w) - \ \mathrm{cap}(\mathbb{B}^n(x,r)) \geq C\, r^{q-1}\, c_{j_0(r)} \left[\int_r^{\infty} t^{\frac{(q-n)}{(q-1)}} \frac{dt}{t} \right]^{1-q}. \tag{3.6}$$

An easy computation shows that the right hand side of (3.6) is equal to

$$C\, r^{q-1}\, c_{j_0(r)}\, r^{n-q} = C\, r^{n-1}\, c_{j_0(r)}.$$

Hence we have that,

$$(q,w) - \ \mathrm{cap}(\mathbb{B}^n(x,r)) \geq C\, c_{j_0(r)}\, r^{n-1}. \tag{3.7}$$

Let E be a subset of $\partial\mathbb{B}^n(0,1)$ and denote the $(n-1)$ dimensional Hausdorff measure of E by

$$\Lambda_{n-1}(E) = \lim_{\delta\to 0} \left[\inf\{\sum_i r_i^{n-1} : E \subset \bigcup \mathbb{B}^n(x_i,r_i),\ 0 < r_i < \delta\} \right],$$

where the infimum is taken over all coverings of E by balls of radii less than δ.

Let $\{\mathbb{B}^n(x_i, r_i): x_i \in E,\ 0 \leq r_i < \delta\}$ be a covering of the set E. If we define by

$$\Lambda^{\delta}_{n-1}(E) = \inf\{\sum_i r_i^{n-1}: E \subset \bigcup \mathbb{B}^n(x_i, r_i),\ 0 < r_i < \delta\},$$

we have that $\Lambda_{n-1}(E) = \lim_{\delta\to 0} \Lambda^{\delta}_{n-1}(E)$. Hence for any of those coverings,

$$\Lambda^{\delta}_{n-1}(E) \leq \sum_i r_i^{n-1} < \sum_i r_i^{n-1}\, c_{j_0(r_i)},$$

provided the r_i are small enough and since $c_{j_0(r)}$ goes to ∞ as r goes to 0. It is clear from our construction that we can always choose our x_i's such that (3.7) holds for all the balls $\mathbb{B}^n(x_i, r_i)$, and thus we have that

$$\Lambda^{\delta}_{n-1}(E) \leq C\{\sum_i (q, w) - \operatorname{cap}(\mathbb{B}^n(x_i, r_i))\}. \tag{3.8}$$

Let f be a bounded quasiregular mapping in $\mathbb{B}^n$ and w be a weight as in Theorem 3.1. Let E be the set of points in $\subset \partial\mathbb{B}^n(0,1)$ for which the nontangential limit of f does not exist. If our weight w will be in $A_q(\mathbb{R}^n)$, by Theorem 2.1 in section §2 we will have that $(q, w) - \operatorname{cap}(E) = 0$.

Thus, by the definition of the weighted variational capacity we have that for any $\tilde{\epsilon} > 0$ we can find a covering of E such that $E \subset \bigcup_i \mathbb{B}^n(x_i, r_i)$ and

$$\sum_i (q, w) - \operatorname{cap}(\mathbb{B}^n(x_i, r_i)) \leq (q, w) - \operatorname{cap}(E) + \tilde{\epsilon}. \tag{3.9}$$

Combining (3.8) and (3.9) we have that

$$\Lambda^{\delta}_{n-1}(E) \leq C\left[(q, w) - \operatorname{cap}(E) + \tilde{\epsilon}\right] = C\,\tilde{\epsilon}$$

and since $\tilde{\epsilon}$ is arbitrary and independent of δ, letting $\delta \to 0$ we will obtain that $\Lambda_{n-1}(E) = 0$.

In our last two paragraphs we have made two assumptions in order to get our conclusion that $\Lambda_{n-1}(E) = 0$ for the set $E \subset \partial\mathbb{B}^n$ where the nontangential limits of the mapping f fail to exist. Namely, the multiplicity function of the mapping f satisfies the following growth condition

$$N(f, B(0, r)) \leq C\,(1 - r)^{-(n-1)} \left(\frac{1}{\log\left(\frac{1}{1-r}\right)}\right)^b$$

and our weight w with the following explicit formula

$$w(x) = \sum_{j=0}^{\infty} c_j \, |1 - |x||^{q-1} \, \aleph_{R_j}(x)$$

is in the Class A_q. Our next result shows that for our second assumption to be true, the constants c_j have an exponential growth to infinity as $j \to \infty$. If that is the case, a simple computation that we will leave to the reader, will show that then we can not ascertain that

$$\int_{\mathbb{B}^n(0,1)} |Df(x)|^q \, w(x) \, dw$$

is finite, and then Theorem 2.1 can not be invoked. This argument shows that we have pushed our approach to the limit in the sense that Theorem 3.11 below is the best result we can obtain following our approach.

In [MV1], it was shown that the positive weight $w(x) = |1-|x||^\alpha$ for $x \in \mathbb{R}^n$ is in the class $A_q(\mathbb{R}^n)$ whenever $q > \alpha + 1$.

The following argument shows that, in some sense, this can not be improved. Namely,

Lemma 3.10. *Let the positive weight w be defined as in Theorem 3.1. If $w(x)$ belongs to the Muckenhoupt class $A_q(\mathbb{R}^n)$ after extending it to be symmetric outside the unit ball of $\mathbb{R}^n$, then the c_j's are equivalent to $c_j = (1 - |x|)^{-\epsilon}$ on each of the rings R_j, for some positive ϵ, and hence our weight w is equivalent to*

$$\sum_{j=0}^{\infty} |1 - |x||^{q-1-\epsilon} \aleph_{R_j}(x),$$

for some positive ϵ.

This shows that if we require the weight $w(x)$ in Theorem 3.1 to be in the class A_q, this forces it to be equivalent to $|1 - |x||^{q-1-\epsilon}$.

Proof of Lemma 3.10. Since the integrals that appear in the definition of the A_q-weights are invariant under rotations for the weights under consideration, it is enough to show that

$$\sup_{\mathbb{B}} \left(\frac{1}{|\mathbb{B}|} \int_{\mathbb{B}} w(x) \, dx \right) \left(\frac{1}{|\mathbb{B}|} \int_{\mathbb{B}} [w(x)]^{\frac{1}{1-q}} \, dx \right)^{q-1} < \infty,$$

where $\mathbb{B}$ is any ball in $\mathbb{R}^n$ whose center falls in the positive real axis. Moreover, we can assume that x_0, the center of the ball, is equal to $(1, 0, \dots, 0)$.

Using polar coordinates, and letting $s = |x|$ we have

$$A=\left(\frac{1}{|\mathbb{B}(x_0,r)|}\int_{\mathbb{B}(x_0,r)} w(x)\,dx\right)\left(\frac{1}{|\mathbb{B}(x_0,r)|}\int_{\mathbb{B}(x_0,r)} [w(x)]^{\frac{1}{1-q}}\,dx\right)^{q-1}$$

$$=\frac{c}{r^{nq}}\left(\int_I\int_{\mathbb{B}(x_0,r)\cap S_s^{n-1}} w(s)\,dS\,ds\right)\left(\int_I\int_{\mathbb{B}(x_0,r)\cap S_s^{n-1}} w(s)^{\frac{1}{1-q}}\,dS\,ds\right)^{q-1},$$

where I is an interval of length equivalent to r on the positive real axis ending at x_0 and c is a constant that depends only on n. Using the fact that

$$\int_{\mathbb{B}(x_0,r)\cap S_s^{n-1}} dS \le c\,r^{n-1}$$

we reduce the problem to one dimension, and thus we need to show that

$$A \le \frac{c}{r^q}\left(\int_I w(s)\,ds\right)\left(\int_I w(s)^{\frac{1}{1-q}}\,ds\right)^{q-1} < \infty.$$

It is clear from the definition of the weight w, that it is enough to show that the above integral is bounded by a constant independent of the length of the interval I whenever I is an interval ending at 1. That is, we need to show that

$$A \le \frac{c}{r^q}\left(\int_{r_0}^1 \sum_{j=0}^{\infty} c_j\,(1-s)^{q-1}\,\aleph_{(1-2^{-j},1-2^{-j-1}]}(s)\,ds\right)$$

$$\left(\int_{r_0}^1\left(\sum_{j=0}^{\infty} c_j\,(1-s)^{q-1}\,\aleph_{(1-2^{-j},12-j-1]}(s)\right)^{\frac{1}{1-q}} ds\right)^{q-1} < \infty,$$

where $r=1-r_0$. Without loss of generality we can assume that $r_0=1-2^{-j_0}$. Letting $t=1-s$ and performing the linear change of variable we need to show that

$$A \le \frac{c}{r^q}\left(\int_0^{1-r_0} \sum_{j=j_0}^{\infty} c_j\,t^{q-1}\,\aleph_{(2^{-j-1},2^{-j}]}(t)\,dt\right)$$

$$\left(\int_0^{1-r_0}\left(\sum_{j=j_0}^{\infty} c_j\,t^{q-1}\,\aleph_{(2^{-j-1},2^{-j}]}(t)\right)^{\frac{1}{1-q}} dt\right)^{q-1}$$

$$\le \frac{c}{r^q}\left(\sum_{j=j_0}^{\infty} c_j\int_{2^{-j-1}}^{2^{-j}} t^{q-1}\,dt\right)\left(\sum_{j=j_0}^{\infty} c_j^{\frac{1}{1-q}}\int_{2^{-j-1}}^{2^{-j}} \frac{dt}{t}\,dt\right)^{q-1}$$

$$=\frac{c}{r^q}\left(\sum_{j=j_0}^{\infty} c_j\,2^{-jq}\,(1-2^{-q})\right)\left(\sum_{j=j_0}^{\infty} c_j^{\frac{1}{1-q}}\right)^{q-1},$$

where the constants c in the above chain of inequalities might be different from line to line, but in any case is independent of the r. We have also used the fact that $\int_{2^{-j-1}}^{2^{-j}} \frac{dt}{t}\, dt = \ln 2$.

It is clear by looking at the two infinite series after the last equality, that in order for A to be bounded by a constant independent of r we need to choose $c_j = \eta^j$ for $1 < \eta < 2^q$. But, this choice of the c_j's on each of the rings R_j is equivalent to say that on those rings $c_j = (1 - |x|)^{-\epsilon}$ for some positive ϵ as we wanted to show. □

Lemma 3.10 shows that we can not construct a weight around the weight $|1 - |x||^{q-1}$ of the form (3.2) which is still in A_q and for which the sequence of c_j's goes to infinity as $j \to \infty$ less than exponentially.

Finally, we will prove the main result in this paper that improves Theorem 4.1 in [KMV] and the results of Martio and Srebro in [MS], on the size of the sets on the boundary of the unit ball of $\mathbb{R}^n$ where the nontangential limits might not exist. We will state our result for bounded quasiregular mappings in $\mathbb{B}^n$, but the same conclusion will follow if we impose a growth condition on the mapping f of the type $|f(x)| \leq C\,(1 - |x|)^{-b}$ for some $0 \leq b < \infty$, as the argument in the proof of Theorem 4.1 in [KMV] shows.

Let us state our main result in this paper.

Theorem 3.11. *Let f be a bounded quasiregular mapping of $\mathbb{B}^n$, and suppose that, for some $0 \leq a < n - 1$,*

$$N(f, B(0, r)) \leq C\,(1 - r)^{-a}$$

for all $0 < r < 1$. Then the set of points $x_0 \in E \subset \partial\mathbb{B}^n(0, 1)$ for which the nontangential limit of f does not exist has Hausdorff dimension less than or equal to a, i.e. $dim_H(E) \leq a$.

Before we pass to the proof of this Theorem, notice that for any $0 < a < n - 1$ we have that $a < \frac{na}{1+a}$, which shows that our result improves the one in [KMV].

Proof of Theorem 3.11. Arguing as in the proof of Theorem 3.1 we arrive at

$$\sum_{j=1}^{\infty} \int_{R_j} |Df(x)|^q\, w(x)\, dx \leq C \sum_{j=1}^{\infty} \Big(\int_{f(R_j)} n(y, f, R_j)\, dy\Big)^{q/n} \Big(\int_{R_j} w(x)^{\frac{n}{n-q}}\, dx\Big)^{\frac{n-q}{n}}.$$

Since

$$\begin{aligned} n(y,f,R_j) &\leq N(f,B(1-2^{-j-1})) \\ &\leq C\,(1-(1-2^{-j-1}))^{-a} \\ &= C\,2^{j\,a}. \end{aligned}$$

For any positive ϵ we choose $w(x) = |1-|x||^{\alpha}$ with $\alpha + 1 < q$. By Lemma 5.1 in [MV1] $w \in A_q(\mathbb{R}^n)$ and we have that in R_j,

$$w(x) \leq C\,2^{-j\,\alpha}.$$

Then we have that,

$$\sum_{j=1}^{\infty} \int_{R_j} |Df(x)|^q\,w(x)\,dx \leq C \sum_{j=1}^{\infty} 2^{j\,a\,\frac{q}{n}}\;2^{-j(\frac{n-q}{n})}\,2^{-j\,\alpha}.$$

Thus, after simplifying we have that

$$\begin{aligned} \sum_{j=1}^{\infty} \int_{R_j} |Df(x)|^q\,w(x)\,dx &\leq C \sum_{j=1}^{\infty} 2^{j(a\,\frac{q}{n} - (\frac{n-q}{n}) - \alpha)} \\ &= C \sum_{j=1}^{\infty} 2^{j((1+a)\,\frac{q}{n} - (\alpha+1))} < \infty, \end{aligned}$$

if and only if $(1+a)\,\frac{q}{n} - (\alpha+1)$ is negative. That is, if and only if $(1+a)\,\frac{q}{n} < \alpha + 1 < q$. Thus, for fixed $0 \leq a < n-1$, let ϵ positive such that $q - 1 > \alpha = (1+a+\epsilon)\,\frac{q}{n} - 1 > (1+a)\,\frac{q}{n} - 1$.

Therefore $\int_{\mathbb{B}^n \setminus B(0,1/2)} |Df(x)|^q\,w(x)\,dx < \infty$, thus $\int_{\mathbb{B}^n} |Df(x)|^q\,w(x)\,dx < \infty$ and the claim will follow.

Applying Theorem 2.1 to each of the components of the bounded quasiregular mapping f, we have that the mapping f has nontangential limits on all radii terminating outside a set of (q,w)-capacity zero, where $1 < q \leq n$ and the weight w as above.

Next, we will show that the set E has Hausdorff dimension less than or equal to $\alpha + n - p$.

We start by observing the following weak type estimate

$$\text{(3.12)} \qquad \Lambda_s^{w,\infty}\left(\{y \in \mathbb{R}^n : M_{s,p}^w f(y) > t\}\right) \leq \frac{c(s,w)}{t^p} \int_{\mathbb{R}^n} |f(x)|^p\,w(x)\,dx,$$

whose proof is actually contained in the proof of Lemma 3.1 in [MV1]. Let u be a function in $C_0^\infty(B(y,R))$, as in [HKM, Lemma 2.30] write

$$|u(y)| \leq c \int_0^\infty \frac{1}{r^n} \int_{B(y,r)} |\nabla u(x)|\, dx\, dr.$$

Inserting $w(x)^{1/p}\, w(x)^{-1/p}$ and applying Hölder's inequality and the A_p condition for the weight w one can easily adapt the proof of [HKM, Lemma 2.30] to obtain

$$|u(x)| \leq c\, R^{1+s/p}\, M^w_{s,p}[|\nabla u|(x)] \tag{3.13}$$

for any $x \in \mathbb{R}^n$ as long as $s > -p$.

Combining (3.12) and (3.13) we obtain

$$\Lambda_s^{w,\infty}\Big(\{x \in \mathbb{B}(y,R)\colon |u(y)| > t\} \Big) \leq c\, \frac{R^{s+p}}{t^p} \int_{\mathbb{B}(y,R)} |\nabla u(x)|^p\, w(x)\, dx. \tag{3.14}$$

The proof of [HKM, Theorem 2.26] can be carried out in the weighted case by using the estimate (3.14), thus the $cap_{p,w}(E) = 0$ implies that $\Lambda_s^{w,\infty}(E) = 0$ for all $s > -p$.

It follows that given any positive ϵ one can cover the set $E \subset \bigcup_i \mathbb{B}(x_i, r_i)$, such that

$$\sum_i r_i^s\, w(\mathbb{B}(x_i, r_i)) < \epsilon.$$

Note that the centers of the balls x_i may be taken on $\partial\mathbb{B}^n$. Using now the special nature of our weight we have that $w(\mathbb{B}(x_i, r_i)) \approx r_i^{n+\alpha}$. Therefore,

$$\sum_i r_i^{n+\alpha+s} < c\,\epsilon.$$

Using for example [HKM, Lemma 2.25] we conclude that the Hausdorff dimension of the set E is less than or equal to $n + \alpha + s$ for any $s > -p$. for any $s > -p$. Thus, $\dim_H(E) \leq n + \alpha - q$.

Finally, choosing $q = n$, since we have that $q - 1 > \alpha = (1 + a + \epsilon)\,\frac{q}{n} - 1 > (1 + a)\,\frac{q}{n} - 1$, then $\alpha = a + \epsilon$ and thus $\dim_H(E) \leq a + \epsilon$. Letting ϵ go to zero, we obtain the desired result and the Theorem is proved. □

The main idea in the proof of Theorem 3.11 is to reduce our situation to the case of a weighted Dirichlet finite quasiregular mapping, and then apply the results of the section §2.

Remarks 3.15. *1) Theorem 3.11 generalizes a result of Koskela, Manfredi and Villamor [KMV] stating that bounded quasiregular mapping f in $\mathbb{B}^n$ satisfying the growth condition $N(f, B(0,r)) \leq C\,(1-r)^{-a}$ for some $0 \leq a < n-1$ on the multiplicity function has nontangential limits at all points on the boundary of the unit ball except possibly on a set whose Hausdorff dimension is strictly less than $\frac{n\,a}{1+a} < n-1$, since for any $0 < a < n-1$ we have that $a < \frac{n\,a}{1+a}$. It also improves on a result of Martio and Srebro [MS], who prove Theorem 3.11 for locally injective bounded quasiregular mappings in the unit ball of $\mathbb{R}^n$. In that paper, Martio and Srebro construct explicit examples of locally injective bounded quasiregular mappings that show that both their and our results are sharp.*

2) We will obtain the same conclusion as in Theorem 3.11 if rather than assuming that the mapping f is bounded and with the same growth condition on its multiplicy function, we will assume that $|f(x)| \leq C\,(1-|x|)^{-b}$ for some $0 < b < \infty$. Namely, the following theorem holds:

Theorem 3.16. *Let f be a bounded quasiregular mapping of $\mathbb{B}^n$, and suppose that, for some $0 \leq a < n-1$,*

$$N(f, B(0,r)) \leq C\,(1-r)^{-a}$$

for all $0 < r < 1$. If $|f(x)| \leq C\,(1-|x|)^{-b}$ for some $0 < b < \infty$, then the set of points $x_0 \in E \subset \partial\mathbb{B}^n(0,1)$ for which the nontangential limit of f does not exist has Hausdorff dimension less than or equal to a, i.e. $dim_H(E) \leq a$.

The proof of this theorem requires, as in the proof of Theorem 4.1 in [KMV], an initial modification of our mapping f by composing it with the mapping $g(x) = x\,|x|^{\epsilon-1}$, and picking $\epsilon > 0$ so that $h(x) = g(f(x))$ satisfies $|h(x)| \leq C\,(1-|x|)^{-s}$, for some $s \geq 0$ with $(1+a+n\,s) < n$.

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