

BOUNDARY VALUES FOR SOME CLASSES OF FUNCTIONS

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ABSTRACT

In this paper we prove Riesz type theorems on uniqueness sets for regular functions in the Dirichlet class and for meromorphic functions in the spherical Dirichlet class.

1. Introduction

In this paper we shall study boundary values for different classes of harmonic, regular and meromorphic functions in the unit disc Δ . We start by introducing the classes of functions that we shall be studying.

DEFINITION A. Let $f(z)$ be a regular function in the unit disc. If

$$\iint_{\Delta} |f'(z)|^2 dx dy < \infty,$$

we say that $f(z)$ is in the Dirichlet class D .

DEFINITION B. Let $f(z)$ be a regular non-constant function in the unit disc. Define

$$n(w) = n(w, \Delta, f)$$

to be the number of roots of the equation $f(z) = w$ in Δ and write

$$p(R) = p(R, \Delta, f) = \frac{1}{2\pi} \int_0^{2\pi} n(Re^{i\theta}) d\theta.$$

Then if there exists a positive number p such that,

$$\frac{1}{\pi} \iint_{\{z: |f(z)| \leq R\}} |f'(z)|^2 dx dy = \int_0^R p(\rho) d\rho^2 \leq pR^2$$

for all positive R , we say that the function $f(z)$ is an *areally mean p -valent function*.

Let us denote by AMP the class of regular areally mean p -valent functions in the unit disc. It is not difficult to see that AMP is not a subset of D and D is not a subset of AMP.

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In this work we shall use the concepts of non-tangential or angular limits and capacity of a set; for their definitions we refer the reader to [3].

The original Fatou theorem states that any bounded harmonic function in the unit disc has radial limits for almost every $e^{i\theta}$ in $\partial\Delta$, and the original M. and F. Riesz theorem states that if a bounded analytic function in the unit disc has the same radial limit in a set of positive Lebesgue measure in $\partial\Delta$, then the function is constant.

Beurling in [1] realized that for some classes of functions the right tool to measure the uniqueness sets is not the Lebesgue measure, but the capacity. He proved the Fatou theorem for the class of regular univalent functions in the unit disc with Lebesgue measure replaced by capacity.

Now we proceed to describe the work done in this paper. In Section 2 we study the following problem. Assume that $f(z)$ is in D , such that $\lim_{z \rightarrow e^{i\theta}} f(z) = \alpha$ non-tangentially for all $e^{i\theta}$ in E a subset of $\partial\Delta$. Carleson in [2] proved that some condition on the limiting value α must be required if we want E to be a set of uniqueness. He constructed a non-constant function in D having the same limiting value in a subset E of $\partial\Delta$ of positive capacity.

What condition must we impose on the limiting value so that if the capacity of E is positive then $f(z)$ is a constant function?

We are going to need the following preliminaries. Let α be a complex number in the plane; we write $\Delta_\rho(\alpha) = \{z : |f(z) - \alpha| < \rho\}$, where $f(z)$ is in the class D . Consider the following area integral:

$$\iint_{\Delta_\rho(\alpha)} |f'(z)|^2 dx dy = A_\rho(\alpha);$$

then we have the following definition.

DEFINITION C. We say that α is an ordinary value of $f(z)$ if,

$$\limsup_{\rho \rightarrow 0} \frac{A_\rho(\alpha)}{\rho^2} < \infty.$$

Tsuji in [9] proved the following result.

THEOREM. Suppose that $f(z)$ is in the class D and that $\lim_{z \rightarrow e^{i\theta}} f(z) = \alpha$ non-tangentially whenever $e^{i\theta}$ is in E , where E is a subset of $\partial\Delta$. If α is an ordinary value of $f(z)$ and E has positive capacity, then $f(z)$ is a constant function.

We prove the following theorem, which extends Tsuji's theorem.

THEOREM 1. Assume that $f(z)$ is in the class D , and suppose that $\lim_{z \rightarrow e^{i\theta}} f(z) = \alpha$ non-tangentially for all $e^{i\theta}$ in E a subset of $\partial\Delta$. If

$$A_\rho(\alpha) = o\left(\rho^2 \log \left[\frac{1}{\rho}\right]\right) \text{ as } \rho \rightarrow 0$$

and E has positive capacity, then $f(z)$ is constant.

It will be interesting to know if our condition on the limiting value α is sharp.

Let $u(z)$ be a harmonic function in the unit disc. Consider the level curves of $u(z)$, denoted by $l(c) = \{z \in \Omega : u(z) = c\}$ for $-\infty < c < \infty$, and let $\Theta(c) = \int_{l(c)} |*du|$ for $-\infty < c < \infty$, where $*du$ is the differential of the harmonic conjugate function of $u(z)$, with the agreement that $\Theta(c) = 0$ if $l(c) = \emptyset$. After these preliminaries we have the following definition.

DEFINITION E. Let $u(z)$ be a harmonic function in the unit disc. Suppose that there exists $a \in u(\Omega)$ such that

$$\int_a^b \frac{dc}{\Theta(c)} < \infty$$

for every b greater than a , and

$$\lim_{b \rightarrow \infty} \int_a^b \frac{dc}{\Theta(c)} = \infty.$$

Under these conditions we say that the function $u(z)$ is a slowly increasing unbounded harmonic function, and we denote this class of functions by S .

Proposition 1 in Section 3 proves that any function in the class S has non-tangential boundary limits almost everywhere in $\partial\Delta$. Then we prove the following theorem.

THEOREM 2. *There exists a function in the class S of maximum growth which does not have non-tangential limits in a set of positive capacity in $\partial\Delta$.*

The definition of a function in the class S of maximum growth can be found in Jenkins and Oikawa [6, p. 58]. This result shows that our Proposition 1 is in some sense best possible.

Let us remark here that if f is in the class AMP, zero free and unbounded then $u = \log |f|$ belongs to the class S ; see Hayman [4]. It is also known [3] that if f is in AMP then it has non-tangential boundary limits at every point in $\partial\Delta$ except possibly in a subset E of $\partial\Delta$ of capacity zero. Thus for the class of functions

$$S_{AMP} = \{u : u(z) = \log |f(z)|, f \in \text{AMP, zero free and unbounded}\},$$

which is a subset of S , a stronger Fatou theorem holds.

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THEOREM 1. *Assume that $f(z)$ is in the class D , and suppose that $\lim_{z \rightarrow e^{i\theta}} f(z) = \alpha$ non-tangentially for all $e^{i\theta}$ in E a subset of $\partial\Delta$. If*

$$A_\rho(\alpha) = o\left(\rho^2 \log \left[\frac{1}{\rho}\right]\right) \text{ as } \rho \rightarrow 0$$

and E has positive capacity, then $f(z)$ is constant.

Proof. For $0 < \delta < 1$, let $C_\delta = \{z : |z| = \delta\}$. By Ω_δ we mean the open region bounded by the two tangents from $z = 1$ to C_δ and by the more distant arc of

C_δ between the points of contact. Let $\Omega_\delta(\theta)$ be the domain Ω_δ rotated through an angle θ around $z = 0$, and consider $\tilde{\Omega}_\delta = \bigcup_{e^{i\theta} \in E} \Omega_\delta(\theta)$. We choose $0 < r_0 < 1$ and let $D_{r_0} = \{z: |z| < r_0\}$; then the domain $\Omega_\delta = D_{r_0} \cup \{\tilde{\Omega}_\delta \cap [c/D_{r_0}]^c\}$ is simply connected, and for any $0 < \delta < 1$, $\partial\Omega_\delta$ is a quasiconformal curve.

Consider now the conformal mapping $\phi_\delta: \Omega_\delta \mapsto \Delta$ with $\phi_\delta(0) = 0$. Then for the map

$$\phi_\delta^{-1} = \tilde{\phi}_\delta: \Delta \mapsto \Omega_\delta,$$

$\partial[\tilde{\phi}_\delta(\Delta)]$ is quasiconformal.

We now take the map

$$\phi'_\delta(z) = \frac{1}{\tilde{\phi}_\delta[1/z]}$$

defined on the exterior of the unit disc. It has a Taylor-series expansion of the form

$$\phi'_\delta(z) = a'_\delta z + b_0 + \frac{b_1}{z} + \dots$$

If the Taylor-series expansion of the function $\tilde{\phi}_\delta(z)$ is $\tilde{\phi}_\delta(z) = a_\delta z + \dots$, it is not difficult to show that $a'_\delta = 1/\bar{a}_\delta$. Since we can choose $a_\delta > 0$, it follows that $a'_\delta = 1/a_\delta$. Thus the function $a_\delta \phi'_\delta(z)$ belongs to Σ (the class of normalized univalent functions in the exterior of the unit disc). By [8, Theorem 9.14, p. 292], since $\partial[a_\delta \phi'_\delta(\partial\Delta)]$ is quasiconformal it follows that $a_\delta \phi'_\delta(z) \in \Sigma[k(\delta)]$ for some $0 < k(\delta) < 1$. Where we say that $f(z) \in \Sigma[k(\delta)]$ if $f(z)$ is in the class Σ and has a quasiconformal extension to the complex plane.

We shall need the following result which can be found in [10].

THEOREM. *Let $g \in \Sigma(k)$, $0 \leq k \leq 1$. If $A \subset \partial\Delta$ is compact then*

$$\text{cap}[g(A)] \leq [\text{cap}(A)]^{1-k}.$$

Without loss of generality we can assume that E is compact. Applying the above result to $A = \phi_\delta(E)$ and $g(z) = a_\delta \phi'_\delta(z) \in \Sigma[k(\delta)]$ we have that

$$a_\delta \text{cap}(E) = \text{cap}(a_\delta E) = \text{cap}[a_\delta \phi'_\delta[\phi_\delta(E)]] \leq [\text{cap}[\phi_\delta(E)]]^{1-k}.$$

Using Schwarz's lemma it is not difficult to show that $a_\delta \geq \frac{1}{2}r_0$. Therefore

$$\frac{1}{2}r_0 \text{cap}(E) \leq [\text{cap}[\phi_\delta(E)]]^{1-k}.$$

Hence if we show that for some δ , $0 < \delta < 1$, $\text{cap}[\phi_\delta(E)] = 0$ we will be done. It is well known [5] that this is equivalent to showing that

$$d_\Delta[\phi_\delta(E), C_{r_0}] = d_{\Omega_\delta}[E, \phi_\delta^{-1}(C_{r_0})] = \infty,$$

where $d_\Delta(E_1, E_2)$ is the extremal length in Δ between the sets E_1 and E_2 . By the comparison principle for extremal lengths and Schwarz's lemma applied to ϕ_δ we obtain that $d_{\Omega_\delta}(E, C_{r_0}) \leq d_{\Omega_\delta}[E, \phi_\delta^{-1}(C_{r_0})]$. Thus, we need to show that for some $\delta > 0$, $d_{\Omega_\delta}(E, C_{r_0}) = \infty$.

Without loss of generality we can assume that $\alpha = 0$. Now we start estimating $d_{\Omega_\delta}(E, C_{r_0})$:

$$d_{\Omega_\delta}(E, C_{r_0}) = \sup_{\rho} \frac{\left\{ \inf_{\gamma} \int_{\gamma} \rho(z) |dz| \right\}^2}{\iint_{\Omega_\delta} \rho^2(z) dx dy},$$

where $\rho(z)$ is a non-negative Borel measurable function in Ω_δ and $\gamma \in \Gamma$ are rectifiable arcs in Ω_δ joining a point of E with a point in C_{r_0} . By our construction of Ω_δ any $\gamma \in \Gamma$ approaches a point of E non-tangentially. Thus $f(\gamma)$ joins 0 with a point in $f(C_{r_0})$.

Since by hypothesis

$$A_\rho(0) = o\left(\rho^2 \log \left[\frac{1}{\rho} \right]\right) \text{ as } \rho \rightarrow 0,$$

for any $\varepsilon > 0$ there exists $\rho_0(\varepsilon) = \rho_0$ such that

$$A_\rho(0) \leq \varepsilon \rho^2 \log \left[\frac{1}{\rho} \right] \tag{2.1}$$

for any $\rho \leq \rho_0$. It is not difficult to show that if $A_\rho(0) = o(\rho^2 \log [1/\rho])$ as $\rho \rightarrow 0$, then for each $\tilde{\rho}_0 < 1$ there exists a constant C_0 which only depends on $\tilde{\rho}_0$ such that

$$A_\rho(0) \leq C_0 \rho^2 \log \left[\frac{1}{\rho} \right] \tag{2.2}$$

for all $\rho \leq \tilde{\rho}_0$.

Let us choose $\tilde{\rho}_0 = e^{-1}$ and take a sequence $\rho_n = e^{-n}$, $n = 1, 2, \dots$ going to zero. For each β small we are going to consider the following non-negative Borel measurable function on Ω_δ . We define

$$\rho^*(z) = \begin{cases} |(f^\beta(z))'| & \text{if } z \in \{|f(z)| < e^{-1}\} \cap \Omega_\delta, \\ 0 & \text{otherwise.} \end{cases}$$

Let us call $\Delta_n^* = \{z: \rho_{n+1} \leq |f(z)| < \rho_n\}$, $n = 1, 2, \dots$. Then,

$$\begin{aligned} \iint_{\Omega_\delta} \rho^{*2}(z) dx dy &\leq \sum_{n=1}^{\infty} \iint_{\Delta_n^*} |(f^\beta(z))'|^2 dx dy, \\ &= \sum_{n=1}^{\infty} \iint_{\Delta_n^*} \beta^2 |f^{\beta-1}(z) f'(z)|^2 dx dy \\ &= \sum_{n=1}^{\infty} \beta^2 \iint_{\Delta_n^*} \frac{|f'(z)|^2}{|f(z)|^{2-2\beta}} dx dy \\ &\leq \beta^2 \sum_{n=1}^{\infty} \frac{1}{\rho_{n+1}^{2-2\beta}} \iint_{\Delta_n^*} |f'(z)|^2 dx dy, \end{aligned}$$

since $\Delta_n^* \subset \Delta_{\rho_n}(0) = \{z: |f(z)| < \rho_n\}$ for any n . Then

$$\iint_{\Omega_\delta} \rho^{*2}(z) dx dy \leq \beta^2 \sum_{n=1}^{\infty} \frac{1}{\rho_{n+1}^{2-2\beta}} \iint_{\Delta_{\rho_n}(0)} |f'(z)|^2 dx dy.$$

For ε fixed, there exists an integer $n_0(\varepsilon) = n_0$ such that $e^{-n_0} \leq \rho_0$. Hence

$$\begin{aligned} \iint_{\Omega_\delta} \rho^{*2}(z) \, dx \, dy &\leq \beta^2 \sum_{n=1}^{n_0-1} \frac{1}{\rho_{n+1}^{2-2\beta}} \iint_{\Delta_{\rho_n}(0)} |f'(z)|^2 \, dx \, dy \\ &\quad + \beta^2 \sum_{n=n_0}^{\infty} \frac{1}{\rho_{n+1}^{2-2\beta}} \iint_{\Delta_{\rho_n}(0)} |f'(z)|^2 \, dx \, dy. \end{aligned}$$

Using inequalities (2.1) and (2.2) in the above sums we obtain

$$\begin{aligned} \iint_{\Omega_\delta} \rho^{*2}(z) \, dx \, dy &\leq \beta^2 \sum_{n=1}^{n_0-1} C_0 \frac{1}{\rho_{n+1}^{2-2\beta}} \rho_n^2 \log \left[\frac{1}{\rho_n} \right] \\ &\quad + \beta^2 \sum_{n=n_0}^{\infty} \varepsilon \frac{1}{\rho_{n+1}^{2-2\beta}} \rho_n^2 \log \left[\frac{1}{\rho_n} \right] \\ &= C_0 \beta^2 e^2 \sum_{n=1}^{n_0-1} e^{-2\beta(n+1)} n \\ &\quad + \varepsilon \beta^2 e^2 \sum_{n=n_0}^{\infty} e^{-2\beta(n+1)} n = \text{I} + \text{II}. \end{aligned}$$

Let us examine II. Now

$$\text{II} = \varepsilon \beta^2 e^2 \sum_{n=n_0}^{\infty} e^{-2\beta(n+1)} n \leq \varepsilon \beta^2 e^2 C_1 \int_0^\infty e^{-2\beta x} x \, dx \leq \varepsilon \beta^2 e^2 \frac{C_2}{\beta^2} = \varepsilon e^2 C_2,$$

where C_2 is a universal constant. Thus we obtain that

$$\iint_{\Omega_\delta} \rho^{*2}(z) \, dx \, dy \leq C_0 e^2 \beta^2 \sum_{n=1}^{n_0-1} e^{-2\beta(n+1)} n + \varepsilon e^2 C_2. \tag{2.3}$$

Now we proceed to estimate $\int_\gamma \rho^*(z) |dz|$ from below for any $\gamma \in \Gamma$:

$$\begin{aligned} \int_\gamma \rho^*(z) |dz| &\geq \sum_{n=1}^{\infty} \int_{\gamma \cap \Delta_n^*} \rho^*(z) |dz| \\ &= \sum_{n=0}^{\infty} \int_{\gamma \cap \Delta_n^*} \beta |f'(z)| |f^{\beta-1}(z)| |dz| \\ &\geq \beta \sum_{n=1}^{\infty} \int_{\gamma \cap \Delta_n^*} \frac{1}{\rho_{n+1}^{1-\beta}} |f'(z)| |dz| \\ &\geq \beta \sum_{n=1}^{\infty} \frac{\rho_n - \rho_{n+1}}{\rho_{n+1}^{1-\beta}} \\ &= \beta \sum_{n=1}^{\infty} \frac{e^{-n} - e^{-n-1}}{e^{-(1-\beta)(n+1)}} \\ &= \beta \sum_{n=1}^{\infty} \frac{1 - e^{-1}}{e^{-(1-\beta)} e^{n\beta}} \\ &= \frac{\beta(1 - e^{-1})}{e^{-(1-\beta)}} \sum_{n=1}^{\infty} e^{-n\beta}. \end{aligned}$$

It is not difficult to show that

$$\sum_{n=1}^{\infty} e^{-n\beta} \geq C_3 \int_0^{\infty} e^{-\beta x} dx = \frac{C_3}{\beta},$$

where C_3 is a universal constant. Thus,

$$\int_{\gamma} \rho^*(z) |dz| \geq C_3 \frac{(1 - e^{-1})}{e^{-(1-\beta)}}. \tag{2.4}$$

Therefore by (2.3) and (2.4),

$$\begin{aligned} d_{\Omega_\beta}(E, C_{r_0}) &\geq \frac{\left\{ \inf_{\gamma} \int_{\gamma} \rho^*(z) |dz| \right\}^2}{\iint_{\Omega_\beta} \rho^{*2}(z) dx dy} \\ &\geq \frac{C_3^2 \left[\frac{1 - e^{-1}}{e^{-(1-\beta)}} \right]^2}{C_0 e^2 \beta^2 \sum_{n=1}^{n_0-1} e^{-2\beta(n+1)} n + e^2 \varepsilon C_2} \end{aligned}$$

letting $\beta \rightarrow 0$ we get that,

$$d_{\Omega_\beta}(E, C_{r_0}) \geq \frac{C_3^2 \left[\frac{1 - e^{-1}}{e^{-1}} \right]^2}{\varepsilon e^2 C_2}.$$

Letting $\varepsilon \rightarrow 0$ we deduce that $d_{\Omega_\beta}(E, C_{r_0}) = \infty$, and the theorem is proved.

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PROPOSITION 1. *Any function $u(z)$ in the class S has non-tangential finite limits for almost every $e^{i\theta}$ in $\partial\Delta$.*

Proof. Since $u(z) \in S$ we have that

$$\mu(a, b) = \int_a^b \frac{dc}{\Theta(c)} < \infty$$

for finite b and $\lim_{b \rightarrow \infty} \mu(a, b) = \infty$. Thus it is clear that for c big enough there exists a subset of $\text{Re}\{w\} = c$ which is omitted by the function $f(z) = u(z) + i * u(z)$, where $*u(z)$ is the harmonic conjugate function of $u(z)$, and this subset has positive measure and therefore positive capacity. A standard result in Nevanlinna theory tells that the function $f(z)$ is in the Nevanlinna class, and functions in the Nevanlinna class have non-tangential limits almost everywhere.

It remains to show that these non-tangential limits are finite almost everywhere. Suppose to the contrary that $f(z)$ has non-tangential limit equal to infinity in a set of positive measure; then by [3, Theorem 2.3] $f(z)$ is a constant function equal to infinity contrary to our assumptions. This concludes the proof of the proposition.

Now we are going to construct a slowly increasing unbounded harmonic function with maximum growth in one direction, which does not have non-tangential limits in a set of positive capacity in $\partial\Delta$.

THEOREM 2. *There exists a function in the class S of maximum growth which does not have non-tangential limits in a set of positive capacity in $\partial\Delta$.*

Proof. We quote the following theorem of [7].

THEOREM A. *There exists a function, analytic and univalent in the unit disc Δ , and a set $E \subset \partial\Delta$ of positive capacity such that $\lim_{r \rightarrow 1} (1-r)^{\frac{1}{2}} |f'(re^{i\theta})| = \infty$ whenever $e^{i\theta}$ belongs to E .*

In [7] the authors construct $f'(z)$ explicitly:

$$f'(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t),$$

where $\mu(t)$, $0 \leq t \leq \pi$, is the monotone non-decreasing singular function associated to a Cantor set E of positive capacity, and they extend $\mu(t)$ to be 0 if $-\pi \leq t \leq 0$ and 1 if $\pi \leq t \leq 2\pi$. From the explicit formula of $f'(z)$ it follows that its real part is positive. If we modify $\mu(t)$ to be $\tilde{\mu}(t) = \mu(t + \frac{1}{2}\pi)$, we have that $\tilde{\mu}(t) = 0$ for $-\frac{1}{2}\pi \leq t \leq \frac{1}{2}\pi$. Then the function

$$\tilde{f}'(z) = \frac{1}{2\pi} \int_{-\pi/2}^{3\pi/2} \frac{e^{it} + z}{e^{it} - z} d\tilde{\mu}(t)$$

has an analytic extension at the point $z = 1$, and $\tilde{f}'(z)$ has the same properties as $f'(z)$. In particular its real part is positive and $\tilde{f}'(z)$ satisfies Theorem A. The function $\log \tilde{f}'(z)$ is well defined and adding a small positive constant, if necessary, we can suppose that $\tilde{f}'(1) \neq 0$. The function $\sin \{\log \tilde{f}'(z)\}$ is bounded since $\text{Re} \{\tilde{f}'(z)\} > 0$, and its derivative is

$$g(z) = \cos \{\log \tilde{f}'(z)\} \frac{\tilde{f}''(z)}{\tilde{f}'(z)}$$

which has an analytic extension at the point $z = 1$ since $\tilde{f}'(z)$ has.

Suppose that

$$\cos \{\log \tilde{f}'(1)\} \frac{\tilde{f}''(1)}{\tilde{f}'(1)} = w_0.$$

Let $N_{w_0}(\varepsilon) = \{w : |w - w_0| < \varepsilon\}$ in the w -plane, then $g^{-1}[N_{w_0}(\varepsilon)]$ is a neighbourhood of $z = 1$ in the z -plane. Therefore there exists a sector S in Δ about $z = 1$ such that $S_\varepsilon = S \cap \{r_0(\varepsilon) < |z| < 1\} \subset g^{-1}[N_{w_0}(\varepsilon)]$ for some $r_0(\varepsilon)$. Take the function

$$h(z) = \log \left[\frac{1}{1-z} \right] + \sin \{\log \tilde{f}'(z)\} + (-1 - w_0)z.$$

We want to show that its real part $u(z) = \text{Re} \{h(z)\} \in S$ has maximum growth in the real direction. We have that

$$h'(z) = \left[\frac{1}{1-z} \right] + \cos \{\log \tilde{f}'(z)\} \frac{\tilde{f}''(z)}{\tilde{f}'(z)} - (1 + w_0);$$

then for $z \in S_\varepsilon$,

$$\left| \left[\frac{1}{1-z} \right] + \cos \{\log \tilde{f}'(z)\} \frac{\tilde{f}''(z)}{\tilde{f}'(z)} - (1 + w_0) \right| \leq \left| \frac{1}{1-z} \right|.$$

Let us point out that the function

$$k(z) = \left[\frac{1}{1-z} \right]$$

is an areally mean $\frac{1}{2}$ -valent function. By [6, p. 39] we have the following inequality:

$$[u(r) - a_0]^2 \leq D_{S_\varepsilon \cap \Omega[a_0, u(r)]}[u] \mu_{S_\varepsilon}[a_0, u(r)], \quad (3.1)$$

where $D_S[u]$ stands for the Dirichlet integral of $u(z)$ over S and $\Omega[a_0, u_r] = \{z: a_0 < u(z) < u(r)\}$. We now proceed to estimate $D_{S_\varepsilon \cap \Omega[a_0, u(r)]}[u]$:

$$D_{S_\varepsilon \cap \Omega[a_0, u(r)]}[u] = \iint_{S_\varepsilon \cap \Omega[a_0, u(r)]} |h'(z)|^2 dx dy$$

since on S_ε , $|h'(z)| \leq |1/(1-z)|$ we have that

$$D_{S_\varepsilon \cap \Omega[a_0, u(r)]}[u] = \iint_{S_\varepsilon \cap \Omega[a_0, u(r)]} \left| \frac{1}{1-z} \right|^2 dx dy.$$

It is not difficult to see that since $\sin\{\log \tilde{f}'(z)\}$ is bounded,

$$\Omega[a_0, u(r)] \subset \tilde{\Omega}[\tilde{a}_0, u(r) + c_0]$$

for some constants \tilde{a}_0, c_0 independent of r , where

$$\tilde{\Omega}[\tilde{a}_0, u(r) + c_0] = \left\{ z: \tilde{a}_0 < \log \left| \frac{1}{1-z} \right| < u(r) + c_0 \right\}.$$

Thus

$$\begin{aligned} D_{S_\varepsilon \cap \Omega[a_0, u(r)]}[u] &\leq \iint_{S_\varepsilon \cap \tilde{\Omega}[\tilde{a}_0, u(r) + c_0]} \left| \frac{1}{1-z} \right|^2 dx dy \\ &\leq \iint_{\tilde{\Omega}[\tilde{a}_0, u(r) + c_0]} \left| \frac{1}{1-z} \right|^2 dx dy. \end{aligned}$$

Since $k(z)$ is an areally mean $\frac{1}{2}$ -valent function, by [6, (3)],

$$D_{S_\varepsilon \cap \Omega[a_0, u(r)]}[u] \leq \pi[u(r) + c_0 - \tilde{a}_0] + \tau.$$

Therefore substituting in (3.1) we have that

$$[u(r) - a_0]^2 \leq \pi[u(r) + c_0 - \tilde{a}_0] \mu_{S_\varepsilon}[a_0, u(r)] + C.$$

Dividing both sides by $[u(r) - a_0]$ we obtain that

$$[u(r) - a_0] \leq \pi \mu_{S_\varepsilon}[a_0, u(r)] + \frac{\pi(a_0 - \tilde{a}_0 + c_0)}{[u(r) - a_0]} \mu_{S_\varepsilon}[a_0, u(r)] + \frac{C}{[u(r) - a_0]}.$$

Now we have that by construction

$$u(r) \geq \log \frac{1}{1-r} - k,$$

and that by [6, (19)]

$$\mu_{S_\varepsilon}[a_0, u(r)] \leq \frac{1}{\pi} \log \frac{1}{1-r} + K'$$

for some universal constant K' . Thus

$$\frac{\pi(a_0 - \tilde{a}_0 + c_0)}{[u(r) - a_0]} \mu_{S_\varepsilon}[a_0, u(r)] \leq K''$$

and therefore

$$\frac{1}{\pi} u(r) - K \leq \mu_{S_\varepsilon}[a_0, u(r)]$$

for some constant K independent of r , and r large enough. Since

$$u(r) = \log \left[\frac{1}{1-r} \right] + \operatorname{Re} \{ \sin \{ \log \tilde{f}'(r) \} \} + \operatorname{Re} \{ (-1 - w_0) \} r$$

and the second and third terms in this expression are bounded, we have that

$$\mu_{S_\varepsilon}[a_0, u(r)] \geq \frac{1}{\pi} \log \frac{1}{1-r} + O(1)$$

as $r \rightarrow 1$. Since $u(z)$ is bounded in $\Delta \setminus S_\varepsilon$, there exists a number b_0 such that $|u(z)| \leq b_0$ for all $z \in \Delta \setminus S_\varepsilon$. Therefore $\Theta_\Delta(c) = \Theta_{S_\varepsilon}(c)$ for all $c \geq b_0$. Thus for $u(r) \geq b_0$ we have that

$$\begin{aligned} \mu_\Delta[a_0, u(r)] &= \mu_\Delta(a_0, b_0) + \mu_{S_\varepsilon}[b_0, u(r)] \\ &\geq \frac{1}{\pi} \log \frac{1}{1-r} + O(1). \end{aligned}$$

Hence

$$\lim_{r \rightarrow 1} \left[\mu_\Delta[a_0, u(r)] - \frac{1}{\pi} \log \frac{1}{1-r} \right] > -\infty,$$

so $u(z)$ attains maximum growth in the direction of the positive real axis.

We have to show now that the function $u(z)$ does not have non-tangential limits in a set of positive logarithmic capacity in $\partial\Delta$. Let us take as our function

$$u(z) = \log \left| \frac{1}{1-z} \right| + \cosh [\arg \tilde{f}'(z)] \sin [\log |\tilde{f}'(z)|] + \operatorname{Re} \{ (-1 - w_0) z \},$$

since $\cosh [\arg \tilde{f}'(z)]$ is greater than or equal to one, and $\lim_{r \rightarrow 1} |\tilde{f}'(re^{i\phi})| \rightarrow \infty$ whenever $e^{i\phi} \in E$. We have that the limit

$$\lim_{r \rightarrow 1} [\cosh [\arg \tilde{f}'(re^{i\phi})] \sin [\log |\tilde{f}'(re^{i\phi})|]]$$

does not exist whenever $e^{i\phi} \in E$. Thus the $\lim_{r \rightarrow 1} u(re^{i\phi})$ does not exist whenever $e^{i\phi} \in E$. Since E has positive capacity we have proved the theorem.

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