SUBMITTED FOR PUBLICATION

REGULARITY THEORY AND TRACES OF $\mathcal{A}$–HARMONIC FUNCTIONS

PEKKA KOSKELA*
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MICHIGAN
ANN ARBOR, MI 48109,

JUAN J. MANFREDI†
DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF PITTSBURGH
PITTSBURGH, PA 15260

AND

ENRIQUE VILLAMOR
DEPARTMENT OF MATHEMATICS
FLORIDA INTERNATIONAL UNIVERSITY
MIAMI, FL 33199

ABSTRACT. In this paper we discuss two different topics concerning $\mathcal{A}$-harmonic functions. These are weak solutions of the partial differential equation

$$\operatorname{div}(\mathcal{A}(x,\nabla u))=0,$$

where $\langle\mathcal{A}(x,\xi),\xi\rangle \approx |\xi|^{p-1}$ for some fixed $p\in(1,\infty)$. First, we present a new approach to the regularity of $\mathcal{A}$-harmonic functions for $p>n-1$. Secondly, we establish results on the existence of nontangential limits for $\mathcal{A}$-harmonic functions in the Sobolev space $W^{1,q}(\mathbb{B})$, for some $q>1$, where $\mathbb{B}$ is a ball in $\mathbb{R}^n$. Here q is allowed to be different from p.

† Research partially supported by NSF grant DMS-9101864 .

§1. Introduction.

In this paper we study weak solutions of the partial differential equation

$$\operatorname{div}(\mathcal{A}(x, \nabla u)) = 0, \tag{1.1}$$

where $\langle \mathcal{A}(x,\xi), \xi\rangle \approx |\xi|^{p-1}$; here $1 < p < \infty$. Solutions of (1.1) are called $\mathcal{A}$-harmonic functions. The prototype of these equations is the p-Laplace equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0.$$

Two topics will be discussed. When $p > n-1$, where n is the dimension of the ambient space, we will present a simple approach to the regularity of weak solutions of (1.1), including the Harnack inequality for nonnegative solutions. We do not use the Moser iteration method as in Serrin [S], but rather exploit the fact that for $p > n-1$, the Sobolev embedding theorem on spheres and the maximum principle is all that is needed. Our key observation is that weak solutions of (1.1) are monotone in a certain weak sense as described in [M]. In the borderline case $p = n$ this program is essentially done in [GLM].

In the second part of the paper we consider the case of general $1 < p < \infty$ and present a generalization and simplification of a number of theorems on the existence of nontangential limits of weak solutions of (1.1) in a ball $\mathbb{B}$ with finite q-Dirichlet integral

$$\int_{\mathbb{B}} |\nabla u|^q \, dx,$$

for some $q > 1$. Note that q is allowed to be different from p. The first result in this direction goes back to Beurling [B], who proved that if u is a harmonic function with finite Dirichlet integral in the unit disk $\mathbb{B}^2$, then the set on $\partial\mathbb{B}^2$ where u fails to have a nontangential limit has logarithmic capacity zero. In Carleson's book [C] Beurling's result is extended to continuous Sobolev functions with finite (even weighted) Dirichlet integral where the logarithmic capacity is replaced by an appropriate Riesz capacity and nontangential limits by radial limits. If the function is, in addition, meromorphic this statement also holds for nontangential limits. Nowadays the existence of radial limits for continuous Sobolev functions is well known in any dimension, see [Mi1] and [R], as well as the fact that in this generality a function need not have a single nontangential limit as it can be seen in [C] for the case $n = p = 2$.

It seems that to pass from radial to nontangential limits we need to work with solutions of elliptic partial differential equations. Notice, however, that for $p > n-1$ monotonicity of the functions in the Sobolev class is all it is needed as is shown in [MV]. The linear case is well understood [Mi2],[NRS]. Our approach to study the boundary behavior is based on the weak Harnack inequality and pointwise regularity of Sobolev functions, and so, we are able to obtain the existence of nontangential limits in the framework of the **nonlinear** potential theory of $\mathcal{A}$-harmonic functions.

In the last section of the paper we apply our methods to the components of quasiregular mappings and improve a result of Martio and Rickman [MR] on the existence of nontangential limits of quasiregular mappings with restricted growth on their counting function.

§2. Regularity of solutions for $p > n-1$.

Let $\alpha(x)$ be a measurable function and β a constant such that for a.e. $x \in \mathbb{R}^n$,

$$0 < \alpha(x) \leq \beta < \infty.$$

Let $\mathcal{A}: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a mapping satisfying the following assumptions:

$$\text{the mapping } x \to \mathcal{A}(x,\xi) \text{ is measurable for all } \xi \in \mathbb{R}^n, \tag{2.1}$$

$$\text{the mapping } \xi \to \mathcal{A}(x,\xi) \text{ is continuous for a.e. } x \in \mathbb{R}^n, \tag{2.2}$$

for all $\xi \in \mathbb{R}^n$ and a.e. $x \in \mathbb{R}^n$

$$\mathcal{A}(x,\xi) \cdot \xi \geq \alpha(x)|\xi|^p \tag{2.3}$$

and

$$|\mathcal{A}(x,\xi)| \leq \beta|\xi|^{p-1}. \tag{2.4}$$

Conditions (2.1) and (2.2) insure that the composed mapping $x \mapsto \mathcal{A}(x, g(x))$ is measurable whenever g is measurable. The degenerate ellipticity of the equation is described by condition (2.3). This is the weakest ellipticity condition that still gives the (weak) maximum principle. Finally, condition (2.4) guarantees that $\mathcal{A}(x, \nabla u)$ can be integrated against functions in $W^{1,p}_{\text{loc}}(\Omega)$ with compact support.

Usually, it is also required that $\alpha(x) > \alpha > 0$ for a. e $x \in \mathbb{R}^n$. We **do not make** this assumption at this moment.

A function $u \in W^{1,p}_{\text{loc}}(\Omega)$, where Ω is a domain of $\mathbb{R}^n$, is $\mathcal{A}$-harmonic if it is a weak solution of equation (1.1); that is, for every $\phi \in C_0^\infty(\Omega)$ we have

$$\int_\Omega \mathcal{A}(x, \nabla u(x)) \cdot \nabla\phi(x)\, dx = 0. \tag{2.5}$$

A simple approximation argument shows that (2.5) then holds for compactly supported functions in $W^{1,p}(\Omega)$.

Recall that a function $u \in W^{1,p}_{\text{loc}}(\Omega)$ is weakly monotone [M] if for every relatively compact subdomain Ω' of Ω and for every pair of constants $m \leq M$ such that

$$(m-u)^+ \in W_0^{1,p}(\Omega')$$

and

$$(u-M)^+ \in W_0^{1,p}(\Omega'),$$

we have

$$m \leq u(x) \leq M \text{ for a.e. } x \in \Omega'. \tag{2.6}$$

Lemma 2.7. *$\mathcal{A}$-harmonic functions are weakly monotone.*

Proof. Using $(u-M)^+$ as a test function in (2.5) we obtain

$$\int_{\Omega'\cap\{x:u(x)>M\}} \mathcal{A}(x,\nabla u(x))\cdot\nabla u(x)\,dx=0.$$

It then follows from (2.3) that

$$\int_{\Omega'\cap\{x:u(x)>M\}} \alpha(x)|\nabla u(x)|^p\,dx\le 0.$$

Therefore, $\nabla u(x)=0$ for a. e. $x\in\Omega'\cap\{x:u(x)>M\}$. This implies that $\nabla(u-M)^+$ vanishes a. e. in Ω', and thus $(u-M)^+$ must be the zero function in Ω'. The proof for the lower bound of u is analogous. $\square$

We can now apply the results (whose proofs only employ the Sobolev embedding theorem on spheres) in [M] on weakly monotone functions to obtain:

Proposition 2.8. *Let u be an $\mathcal{A}$-harmonic function in a domain Ω.*

Suppose that $n-1<p<n$. Then u is locally bounded with

$$(\text{ess-osc}_{B_r(x_0)}u)^p\le C(n,p)r^p \fint_{B(x_0,2r)}|\nabla u|^p\,dx \tag{2.9}$$

whenever $B(x_0,2r)\subset\Omega$.

Moreover, u is continuous in Ω, except for a set of p-capacity zero. In the case $p=n$, u is continuous in Ω. Moreover,

$$(\text{osc}_{B(x_0,r)}u)^n\le C(n)(\log(\frac{R}{r}))^{-1}\int_{B(x_0,R)}|\nabla u|^n\,dx \tag{2.10}$$

whenever $B(x_0,R)\subset\Omega$ and $r<R$.

We observe here that in the case $p<n$ there are examples of weakly monotone functions that are not continuous (see [M]).

On the other hand, $\mathcal{A}$-harmonic functions are continuous in the uniformly elliptic case

$$0<\alpha\le\alpha(x), \tag{2.11}$$

since they are always Hölder continuous. This was proved, together with the Harnack inequality for nonnegative $\mathcal{A}$-harmonic functions, by Serrin [S] for the complete range of p's, $1<p<\infty$.

We now give a simple proof of the Harnack inequality for nonnegative $\mathcal{A}$-harmonic functions, $n-1<p\le n$, assuming the uniform ellipticity condition (2.11).

First we need a lemma which is trivial for continuous monotone functions but requires some work for weakly monotone functions.

Lemma 2.12. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, $p > 1$, be a nonnegative weakly monotone function and $\phi\colon \mathbb{R}^+ \to \mathbb{R}$ be C^1-smooth, satisfying $\phi' > 0$ and onto, such that $\nabla(\phi \circ u) \in L^p_{\mathrm{loc}}(\Omega)$. Then $\phi \circ u$ is weakly monotone.*

Proof. Let Ω' be a relatively compact subdomain of Ω and $M \in \mathbb{R}$ such that $(\phi(u) - M)^+ \in W^{1,p}_0(\Omega')$. We have to show that

$$\phi(u(x)) \leq M \text{ for a. e. } x \in \Omega'. \tag{2.13}$$

Choose $\alpha \in \mathbb{R}^+$ such that $\phi(\alpha) = M$ and set $u_k = \min(u, k)$ for a positive integer k. Then, $u_k \in W^{1,p}_0(\Omega')$ and $0 \leq u_k \leq u$. By the monotonicity of ϕ we conclude that

$$0 \leq (\phi(u_k) - \phi(\alpha))^+ \leq (\phi(u) - \phi(\alpha))^+,$$

which implies $(\phi(u_k) - \phi(\alpha))^+ \in W^{1,p}_0(\Omega')$. Select a sequence of nonnegative functions $\phi_m \in C^\infty_0(\Omega') \cap W^{1,p}_0(\Omega')$ such that $\phi_m \to (\phi(u_k) - \phi(\alpha))^+$ in $W^{1,p}(\Omega')$. Note that since u_k is bounded, it is possible to choose this sequence so that it is uniformly bounded in Ω'. It then easily follows that $\phi^{-1} \circ \phi_m \to \phi^{-1}\left((\phi(u_k) - \phi(\alpha))^+\right)$ in $W^{1,p}(\Omega')$. Note that $\phi^{-1} \circ \phi_m \geq \phi^{-1}(0)$ and $\phi^{-1} \circ \phi_m - \phi^{-1}(0)$ has compact support in Ω'. We conclude that

$$\phi^{-1}\left((\phi(u_k) - \phi(\alpha))^+\right) - \phi^{-1}(0) \in W^{1,p}_0(\Omega').$$

At this point observe that for some constant $C(\phi, \alpha, k) > 0$ depending possibly on ϕ, α and k, we have

$$\phi^{-1}\left((\phi(u_k) - \phi(\alpha))^+\right) - \phi^{-1}(0) \geq C(\phi, \alpha, k)\,(\phi(u_k) - \phi(\alpha))$$

whenever $u_k > \alpha$. By possibly modifying $C(\phi, \alpha, k) > 0$ we obtain

$$0 \leq C(\phi, \alpha, k)(u_k - \alpha)^+ \leq \phi^{-1}\left((\phi(u_k) - \phi(\alpha))^+\right) - \phi^{-1}(0).$$

Therefore, we conclude that $(u_k - \alpha)^+ \in W^{1,p}(\Omega')$. Letting $k \to \infty$ we deduce that $(u - \alpha)^+ \in W^{1,p}(\Omega')$. Since u is weakly monotone we must have $u(x) \leq \alpha$ for a. e. $x \in \Omega'$, which implies (2.13). The proof for the lower bound of $\phi \circ u$ is analogous. $\square$

Lemma 2.14. *Assume that (2.11) holds and that $n - 1 < p \leq n$. Let $u > \epsilon > 0$ be $\mathcal{A}$-harmonic in Ω. Then $\log u$ belongs to $W^{1,p}_{\mathrm{loc}}(\Omega)$ and is weakly monotone.*

Proof. We start with a well known trick. Let $B(x_0, 2r) \Subset \Omega$ and let $\eta \in C^\infty_0(B(x_0, 2r))$ be a nonnegative test function. Using the function $\eta^p u^{1-p}$ in (2.5) and applying (2.11) we get

$$\int_{B(x_0,2r)} |\nabla \log u|^p \, dx \leq c(n, p, \frac{\beta}{\alpha}) \int_{B(x_0,2r)} |\nabla \eta|^p \, dx.$$

Choosing $\eta(x) = 1$ for $x \in B(x_0, r)$ we get

$$\int_{B(x_0,r)} |\nabla \log u|^p \, dx \leq c(n, p, \frac{\beta}{\alpha}) r^{n-p}. \tag{2.15}$$

The conclusion now follows from the previous Lemma. $\square$

Theorem 2.16. *Assume that the uniform ellipticity condition (2.11) is satisfied and that we are in the range $n-1 < p \le n$. Let $u \ge 0$ be a nonnegative $\mathcal{A}$-harmonic function and B a ball such that $4B \subset \Omega$. Then,*

$$\sup_{x\in B} u(x) \le C(n,p,\frac{\beta}{\alpha}) \inf_{x\in B} u(x), \tag{2.17}$$

where $C(n,p,\frac{\beta}{\alpha})$ is a constant depending only on n, p and $\frac{\beta}{\alpha}$.

Proof. Let $\epsilon > 0$ and consider the $\mathcal{A}$-harmonic function $v = u + \epsilon$. It is enough to prove estimate (2.17) for v with a constant $C(n,p,\frac{\beta}{\alpha})$ independent of ϵ. By Lemma (2.14), $\log v$ is weakly monotone and

$$\int_{2B} |\nabla \log v|^p \, dx \le c(n,p,\frac{\beta}{\alpha})|B|^{1-p/n}. \tag{2.18}$$

By virtue of Proposition 2.8, $\log v$ is bounded and

$$(\text{ess-osc}_B(\log v))^p \le C(n,p)|B|^{p/n} \fint_{2B} |\nabla \log v|^p \, dx. \tag{2.19}$$

From (2.18) and (2.19) it follows that

$$\text{ess-osc}_B(\log v) \le C(n,p,\frac{\beta}{\alpha})$$

and exponentiating we obtain (2.17) $\square$

§3. Existence of Nontangential Limits

In this section we prove results on boundary limits for Dirichlet finite $\mathcal{A}$-harmonic functions.

First we show that p-Dirichlet finite $\mathcal{A}$-harmonic functions u, defined in the unit ball $\mathbb{B}^n$ of $\mathbb{R}^n$, have nontangential limits everywhere on the boundary of the unit ball except possibly on a set E of Bessel $B_{1,p}$-capacity zero, $1 < p \leq n$. In this work we will use the Bessel $B_{1,p}$-capacity for technical reasons. We refer the reader to the book by Ziemer [Z] for the definition and properties of the Bessel capacity $B_{1,p}$. At this point, we would like to remark that all the p-capacities are equivalent in the sense that a set with one of the standard p-capacities zero will have all the other p-capacities zero. Thus, in the rest of the paper we will say p-capacity zero without specifying the capacity that we are using.

The case $p > n$ is not interesting because then u is continuous up to the boundary by the Sobolev embedding theorem. Recall that an $\mathcal{A}$-harmonic function u of $\mathbb{B}^n$ is continuous in $\mathbb{B}^n$.

Theorem 3.1. *Let u be an $\mathcal{A}$-harmonic function in the unit ball $\mathbb{B}^n$ of $\mathbb{R}^n$ (no restriction on the type of $\mathcal{A}$). If $\int_{\mathbb{B}^n} |\nabla u(x)|^p \, dx < \infty$ for some $1 < p \leq n$, then the function u has nontangential limits on all radii terminating outside a set of p-capacity zero.*

The proof of theorem 3.1 is based on the following two well known lemmas.

Lemma 3.2. *[Z, Theorem 3.3.3] Let $u \in W^{1,p}(\mathbb{R}^n)$, $1 < p < n$. Then*

$$\lim_{r \to 0} \fint_{B(x,r)} |u(y) - u(x)|^p \, dy = 0 \tag{3.3}$$

except for x in a set $E \subset \mathbb{R}^n$ of p-capacity zero.

Lemma 3.4. *[HKM, Theorem 3.34] Let u be an $\mathcal{A}$-harmonic function in $\mathbb{B}^n$, and fix $p > 0$. Then, there exists a constant C such that for each ball $B = B(x,r) \subset \mathbb{B}^n$ and all $a \in \mathbb{R}$*

$$\sup_{\frac{1}{2}B} |u(y) - a| \leq C \, (\fint_B |u(y) - a|^p \, dy)^{1/p},$$

where $\frac{1}{2}B = B(x, r/2)$.

Proof of Theorem 3.1. By [R] and [V, 16.8] (see also [MV]) we may assume that $1 < p < n$. Since $\int_{\mathbb{B}^n} |\nabla u(x)|^p \, dx < \infty$, it follows from the Poincaré inequality that $u \in W^{1,p}(\mathbb{B}^n)$. Hence, by standard extension theorems, we may assume that $u \in W^{1,p}(\mathbb{R}^n)$. We show that u has a nontangential limit for each $x \in \partial\mathbb{B}^n$ for which (3.3) holds. The claim then follows from Lemma 3.2.

Fix an $x \in \partial\mathbb{B}^n$ for which (3.3) holds. Now, for any $0 < t < 1$, Lemma 3.4 gives

$$|u(t\frac{x}{|x|}) - u(x)| \le C\, (\fint_{B_{x,t}} |u(y) - u(x)|^p\, dy)^{1/p}$$

where $B_{x,t} = B(t\frac{x}{|x|}, (1-t)/2)$, and thus

$$|u(t\frac{x}{|x|}) - u(x)| \le C'\, (\fint_{B(x,2(1-t))} |u(y) - u(x)|^p\, dy)^{1/p}.$$

We conclude from (3.3) that u has a radial limit at x.

For a point $x \in \partial\mathbb{B}^n$ we denote by $C(x)$ the Stolz cone at x with a fixed given aperture. Then we can find a constant $c_n \ge 1$, depending only on the aperture and n, such that for all $y \in C(x)$

$$|y - x| \le c_n(1 - |y|).$$

Next, we show that the radial limit at x we found above, is attained through any standard Stolz cone $C(x)$; thus this radial limit is actually a nontangential limit. Pick $x_1 \in C(x)$. Then, we have that

$$B(x_1, (1 - |x_1|)/2) \subset B(x, (c_n + \frac{1}{2})(1 - |x_1|))$$

for each $x_1 \in C(x)$, and hence, by Lemma 3.4,

$$\begin{aligned}|u(x_1) - u(x)| &\le C\, (\fint_{B(x_1,(1-|x_1|)/2)} |u(y) - u(x)|^p\, dy)^{1/p} \\ &\le C'\, (\fint_{B(x,(c_n+\frac{1}{2})(1-|x_1|))} |u(y) - u(x)|^p\, dy)^{1/p}.\end{aligned}$$

The claim follows by applying (3.3). □

In Theorem 3.1, $\mathcal{A}$-harmonicity was not essential but merely the version of the weak Harnack inequality (Lemma 3.4) satisfied by the solutions to a large class of elliptic nonlinear P.D.E.'s.

We continue by showing that Dirichlet finite $\mathcal{A}$-harmonic functions also possess "tangential" limits. By saying *tangential limits of order* $\gamma \ge 1$, we mean that u has a limit as $y \to w \in \partial\mathbb{B}^n$ with $(1 - |y|) \ge C\,|y - w|^\gamma$ for some constant C.

Theorem 3.5. *Assume that u is an $\mathcal{A}$-harmonic function in $\mathbb{B}^n$, and suppose that*

$$\int_{\mathbb{B}^n} |\nabla u|^p\, dx < \infty,$$

for some $1 < p \le n$. If $1 < p < n$ and $1 < \gamma \le \frac{n-1}{n-p}$, then u has tangential limits of order γ outside a set of vanishing $\mathcal{H}^\lambda$-Hausdorff measure, $\lambda = \gamma(n-p)$. If $p = n$, then there is a set of vanishing n-capacity such that u has tangential limits of any order outside this set.

Remark 3.6. For harmonic functions, this result is due to Mizuta [Mi3]. Mizuta also allows weights of the type $w(x) = (1-|x|)^\alpha$, $\alpha < p-1$. We will comment on results with weights of this type below.

We record the following lemma, see [Z, Corollary 3.2.3], [Mi3, Lemma 2].

Lemma 3.7. *Suppose that $u \in W^{1,p}(\mathbb{B}^n)$ for some $1 \le p < \infty$, and let $0 \le \lambda \le n-1$. Then for any $w \in \partial\mathbb{B}^n$, except for w in a set $E \subset \partial\mathbb{B}^n$ of $\mathcal{H}^\lambda$-Hausdorff measure zero*

$$\lim_{r\to 0} \frac{\int_{B(w,r)\cap\mathbb{B}^n} |\nabla u(y)|^p\, dy}{r^\lambda} = 0.$$

Moreover, for any $w \in \partial\mathbb{B}^n$, except for w in a set $E \subset \partial\mathbb{B}^n$ of n-capacity zero,

$$\lim_{r\to 0}(\log\frac{1}{r})^{n-1} \int_{B(w,r)\cap\mathbb{B}^n} |\nabla u(y)|^p\, dy = 0.$$

Proof of Theorem 3.5. Let first $1 < p < n$. Assume that u has a radial limit at $w \in \partial\mathbb{B}^n$, and suppose that

$$\lim_{r\to 0} \frac{\int_{B(w,r)\cap\mathbb{B}^n} |\nabla u(y)|^p\, dy}{r^{\gamma(n-p)}} = 0.$$

By [R] and Lemma 3.7, this holds for every $w \in \partial\mathbb{B}^n$, except possibly for a set of $\gamma(n-p)$-Hausdorff measure zero. Fix $y \in \mathbb{B}^n$ with $|w-y| < 1$, and set $B_y = B(y,(1-|y|)/2)$. Then, by Lemma 3.4,

$$\sup_{\frac{1}{2}B_y} |u(z) - u_{B_y}| \le C(\fint_{B_y} |u(x) - u_{B_y}|\, dx)^{1/p}.$$

Here and in what follows, all the constants will be denoted by C. Using Poincaré's inequality we conclude that

$$\begin{aligned}\sup_{\frac{1}{2}B_y} |u(z) - u_{B_y}| &\le C(\fint_{B_y} |u(x) - u_{B_y}|\, dx)^{1/p} \\ &\le C(1-|y|)^{\frac{p-n}{p}} (\int_{B_y} |\nabla u(x)|^p\, dx)^{1/p}. \end{aligned} \tag{3.8}$$

Hence

$$\operatorname{osc}_{\frac{1}{2}B_y} u \le C(1-|y|)^{\frac{p-n}{p}} (\int_{B_y} |\nabla u(x)|^p\, dx)^{1/p}. \tag{3.9}$$

Since $B_y \subset B(w,r)$ for $r = 2|w-y|$, we arrive at

$$\begin{aligned} \operatorname{osc}_{\frac{1}{2}B_y} u &\le C(1-|y|)^{\frac{p-n}{p}} \Big(\int_{B_y} |\nabla u(x)|^p\, dx\Big)^{1/p} \\ &\le C(1-|y|)^{\frac{p-n}{p}} \Big(\int_{B(w,r)\cap\mathbb{B}^n} |\nabla u(x)|^p\, dx\Big)^{1/p}. \end{aligned} \tag{3.10}$$

Let x_0 be the point on the radius to w belonging to $S^{n-1}(w,|w-y|)$. By the above argument, (3.9) holds with y replaced by y', for each $y' \in S^{n-1}(w,|w-y|)\cap\mathbb{B}^n$. Pick points $y = y_0, \cdots, y_k = x_0$ belonging to $S^{n-1}(w,|w-y|)$ so that the balls $\frac{1}{2}B_{y_j}$ form a chain of balls. By a chain of balls we mean that the intersection of any pair of consecutive balls is non-empty, the diameters increase in a geometric sequence, and only consecutive balls overlap. Then (3.10) yields

$$\begin{aligned} |u(x_0)-u(y)| &\le \sum_{j=1}^{k} |u(y_j)-u(y_{j-1})| \\ &\le C\sum_{j=1}^{k}(1-|y_j|)^{\frac{p-n}{p}}\Big(\int_{B_{y_j}} |\nabla u(x)|^p\, dx\Big)^{1/p} \\ &\le C\,(1-|y|)^{\frac{p-n}{p}}\Big(\int_{B(w,r)} |\nabla u(x)|^p\, dx\Big)^{1/p}. \end{aligned}$$

Recall that $u(x_0)$ tends to a limit as $x_0 \to w$ radially. Since

$$\lim_{r\to 0} \frac{\int_{B(w,r)} |\nabla u(x)|^p\, dx}{r^{\gamma(n-p)}} = 0,$$

and $r = 2|w-y|$, the claim follows (we require that $(1-|y|) \ge C\,|w-y|^{\gamma}$).

Let us finally consider the case $p = n$. Then (3.9) reads

$$\operatorname{osc}_{\frac{1}{2}B^y} u \le C\Big(\int_{B_y} |\nabla u(x)|^n\, dx\Big)^{1/n},$$

and by a chaining argument as above we obtain

$$\begin{aligned} |u(x_0)-u(y)| &\le \sum_{j=1}^{k} |u(y_j)-u(y_{j-1})| \\ &\le C\sum_{j=1}^{k}\Big(\int_{B_{y_j}} |\nabla u(x)|^n\, dx\Big)^{1/n} \\ &\le C\,k^{n-1}\Big(\int_{\bigcup_{j=1}^{k} B_{y_j}} |\nabla u(x)|^n\, dx\Big)^{1/n} \\ &\le Ck^{n-1}\Big(\int_{B(w,r)\cap\mathbb{B}^n} |\nabla u(x)|^n\, dx\Big)^{1/n}. \end{aligned}$$

Here we used the fact that no point in $\mathbb{B}^n$ belongs to more than $C(n)$ of the balls B_{y_j} associated with our chain. We leave it to the reader to check that we can take $k \leq C\log(\frac{1}{1-|y|})$. The claim then follows as in the first part of the proof applying Lemma 3.7 and [R]. $\square$

Theorem 3.11. *Let u be $\mathcal{A}$-harmonic in $\mathbb{B}^n$. If*

$$\int_{\mathbb{B}^n} |\nabla u(x)|^p (1-|x|)^\alpha \, dx < \infty,$$

where $1 < p < n$ and $p - n < \alpha < p - 1$, then u has tangential limits of order $1 < \gamma \leq \frac{n-1}{n-p+\alpha}$ outside a set of vanishing $\gamma(n-p+\alpha)$-Hausdorff measure on $\partial\mathbb{B}^n$.

Proof. We replace dx by $w(x)\,dx$, $w(x) = (1-|x|)^\alpha$, in Lemma 3.7. Then (3.9) reads

$$\begin{aligned}
\operatorname{osc}_{B^y} u &\leq C(1-|y|)^{\frac{p-n}{p}} \Big(\int_{B_y} |\nabla u(x)|^p \, dx\Big)^{1/p} \\
&\leq C(1-|y|)^{\frac{p-n-\alpha}{p}} \Big(\int_{B_y} |\nabla u(x)|^p \, w(x)\, dx\Big)^{1/p} \\
&\leq C(1-|y|)^{\frac{p-n-\alpha}{p}} \Big(\int_{B(w,r)\cap\mathbb{B}^n} |\nabla u(x)|^p \, w(x)\, dx\Big)^{1/p}.
\end{aligned}$$

The existence of radial limits follows from [Mi2], and then the rest of the argument is analogous to the proof of Theorem 3.5. $\square$

§4. Nontangential limits for Quasiregular Mappings.

Let $W^{1,n}_{\text{loc}}(\mathbb{B}^n)$ denote the local Sobolev space of functions in $L^n_{\text{loc}}(\mathbb{B}^n)$ whose distributional derivatives belong to $L^n_{\text{loc}}(\mathbb{B}^n)$. Consider a mapping

$$f: \mathbb{B}^n \to \mathbb{R}^n$$

whose coordinate functions belong to $W^{1,n}_{\text{loc}}(\mathbb{B}^n)$. Denote by $J_f(x)$ the Jacobian determinant $\det(Df(x))$. For a.e. $x \in \mathbb{B}^n$ the *dilatation* of f is defined by

$$K(x) = \frac{|Df(x)|^n}{J_f(x)},$$

and it satisifies $K(x) \geq 1$. If $K(x) \in L^\infty(\mathbb{B}^n)$, then f is said to be a quasiregular mapping.

It is well known, see [HKM], that the coordinate functions of a quasiregular mapping are $\mathcal{A}$-harmonic functions of type n. Therefore, all the results of the previous section apply to quasiregular mappings.

Let E be a subset of the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$. We define $n(y; f, E) = \text{card}\{x \in E: f(x) = y\}$, and $N(f, E) = \sup_{y \in E} n(y; f, E)$. $N(f, E)$ is called the *multiplicity function* of f. In this section we prove that a certain restriction on the growth of the multiplicity function of f implies the existence of nontangential limits. More precisely, we establish:

Theorem 4.1. *Let f be a quasiregular mapping of $\mathbb{B}^n$, and suppose that, for some $0 \leq a < n - 1$, $N(f, B(0, r)) \leq C(1 - r)^{-a}$ for all $0 < r < 1$.*

If $|f(x)| \leq C(1 - |x|)^{-b}$ for some $0 \leq b < \infty$, then f has nontangential limits at all points on the boundary of the unit ball except possibly on a set of p-capacity zero, for any $1 < p < \frac{n}{1+a}$.

Before proving Theorem 4.1, let us make some remarks.

Remarks 4.2. 1) Nontangential limits at all points on the boundary of the unit ball except possibly on a set of p-capacity zero is always better than nontangential limits at all points on the boundary of the unit ball except possibly on a set of $(n-1)$-Hausdorff measure zero, as long as $p > 1$. Because the Hausdorff dimension of a set of vanishing p-capacity is at most $n - p$. Hence Theorem 4.1 extends a result of Martio and Rickman [MR] according to which a bounded quasiregular mapping f of $\mathbb{B}^n$ satisfying the growth condition of Theorem 4.1 on the multiplicity function has nontangential limits at all points on the boundary of the unit ball except possibly on a set of $(n-1)$-Hausdorff measure zero.

2) There are analytic functions of arbitrarily slow growth without a single radial limit; see [ML]. Thus some condition on f is needed.

3) Notice that b plays no role. If f is quasiconformal, then $|f(x)| \leq C(1 - |x|)^{-b}$, for some $b = b(n, K) < \infty$, and we may take $a = 0$. Therefore we recover

that quasiconformal mappings have nontangential limits everywhere except possibly on a set $E \subset \partial\mathbb{B}^n$ of p-capacity zero. As is well known, this holds for $p = n$, see [V]. This appears to indicate that our conclusion is fairly sharp.

4) The main idea in the proof of Theorem 4.1 is to reduce our situation to the case of a Dirichlet finite quasiregular mapping, and then apply results of the previous section. Notice that f also has some "tangential" limits as seen by applying Theorem 3.5 instead of Theorem 3.1 at the last step of the proof. The proof of Theorem 4.1 shows that $|Dg(x)|^q$ is integrable for all $0 < q < 1$ for a bounded quasiregular mapping g of $\mathbb{B}^n$. For a related result on $\mathcal{A}$-harmonic functions see [KK].

Proof of Theorem 4.1. Fix $1 < p < \frac{n}{1+a}$. For $\epsilon > 0$ define $g(x) = x|x|^{\epsilon-1}$. Pick $\epsilon > 0$ so that $h(x) = g(f(x))$ satisfies

$$|h(x)| \leq C(1-|x|)^{-s},$$

for some $s \geq 0$ with $p < \frac{n}{1+a+s}$.

Then, h is a quasiregular mapping of the unit ball $\mathbb{B}^n$. It easily follows that if $h(x) \to z$ as $x \to w$ along $\Omega \subset \mathbb{B}^n$, then also f has a limit as $x \to w$ through Ω. Write $R_j = \{x : 1 - 2^{-j} < |x| \leq 1 - 2^{-j-1}\}$. Now

$$\sum_{j=1}^{\infty} \int_{R_j} |Dh(x)|^p \, dx \leq C \sum_{j=1}^{\infty} \Big(\int_{R_j} |Dh(x)|^n \, dx \Big)^{p/n} 2^{-j(\frac{n-p}{n})},$$

and by a change of variables [BI, 8.3] we arrive at

$$\sum_{j=1}^{\infty} \int_{R_j} |Dh(x)|^p \, dx \leq C \sum_{j=1}^{\infty} \Big(\int_{h(R_j)} n(y,h,R_j) \, dy \Big)^{p/n} 2^{-j(\frac{n-p}{n})}.$$

Since $n(y,h,R_j) \leq N(f, B(1-2^{-j-1}))$ and since $h(R_j) \subset B(0, C2^{sj})$,

$$\sum_{j=1}^{\infty} \int_{R_j} |Dh(x)|^p \, dx \leq C \sum_{j=1}^{\infty} 2^{j(a+s)\frac{p}{n}} \, 2^{-j(\frac{n-p}{n})} < \infty,$$

because $p < \frac{n}{1+a+s}$.

Therefore $\int_{\mathbb{B}^n \setminus B(0,1/2)} |Dh(x)|^p \, dx < \infty$, and thus $\int_{\mathbb{B}^n} |Dh(x)|^p \, dx < \infty$. Applying Theorem 3.1 to the coordinate functions of h we obtain the desired conclusion. □

References

[B] Beurling, A., *Ensembles exceptionnels*, Acta Mathematica **72** (1940), 1–13.

[BI] Bojarski, B. and Iwaniec, T., *Analytical foundations of the theory of quasiconformal mappings in* $\mathbb{R}^n$., Ann. Acad. Sci. Fenn. Ser. A I Math. **8** (1983), 257–324.

[C] Carleson, L., *Selected Problems in Exceptional Sets*, vol. 13, Van Nostrand Mathematical Studies, 1967.

[GLM] Granlund, S., Lindqvist, P. and Martio, O., *Conformally invariant variational integrals*, Trans. Amer. Math. Soc. **277** (1983), 43–73.

[HKM] Heinonen, J., Kilpeläinen, T. and Martio, O., *Nonlinear Potential Theory*, Oxford University Press, 1993.

[KK] Kilpeläinen, T. and Koskela, P., *Global integrability of gradients of solutions to partial differential equations*, Nonlinear Anal. (to appear).

[ML] MacLane, G. R., *Holomorphic functions, of arbitrarily slow growth, without radial limits*, Michigan Math. J. **9** (1962), 21–24.

[M] Manfredi, J. J., *Monotone Sobolev functions* (to appear).

[MV] Manfredi, J. J. and Villamor, E., *Traces of monotone Sobolev functions* (to appear).

[MR] Martio, O. and Rickman, S., *Boundary behavior of quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A I Math. **507** (1972), 1–17.

[Mi1] Mizuta, Y., *Existence of various boundary limits of Beppo Levi functions of higher order*, Hiroshima Math. J. **9** (1979), 717–745.

[Mi2] Mizuta, Y., *Boundary behavior of p-precise functions on a half space of* $\mathbb{R}^n$, Hiroshima Math. J. **18** (1988), 73–94.

[Mi3] Mizuta, Y., *On the boundary limits of harmonic functions with gradient in* L^p, Ann. Inst. Fourier **34** (1984), 99–109.

[NRS] Nagel, A., Rudin, W. and Shapiro, J. H., *Tangential boundary behavior of functions in Dirichlet-type spaces*, Ann. of Math. **116** (1982), 331–360.

[R] Reshethnyak, Yu. G., *Boundary behavior of functions with generalized derivatives*, Sib. Math. J. **13** (1972), 285–290.

[S] Serrin, J., *Local behavior of solutions of quasi-linear equations*, Acta Math. **111** (1964), 247–302.

[V] Vuorinen, M., *Conformal Geometry and Quasiregular Mappings*, vol. 1319, Lecture Notes Mathematics, Springer Verlag, 1988.

[Z] Ziemer, W., *Weakly Differentiable Functions*, vol. 120, Graduate Texts in Mathematics, Springer Verlag, 1989.