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TRACES OF MONOTONE SOBOLEV FUNCTIONS

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Abstract. In this paper we prove that if $u\colon \mathbb{B}^n \to \mathbb{R}$, where $\mathbb{B}^n$ is the unit ball in $\mathbb{R}^n$, is a monotone function in the Sobolev space $W^{1,p}(\mathbb{B}^n)$, and $n-1 < p \leq n$, then u has nontangential limits at all the points of $\partial\mathbb{B}^n$ except possibly on a set of p-capacity zero.

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§1. Introduction.

A classical Theorem of Fatou states that a bounded harmonic function in the unit ball $\mathbb{B}^n$ has nontangential limits at almost everywhere $x \in \partial\mathbb{B}^n$ with respect to Lebesgue measure. This theorem has been extended to solutions of a large class of elliptic partial differential equations, including some nonlinear types. See [CFMS], [FGMS] and [MW].

The validity of its extension to the higher dimensional analogues of holomorphic functions, quasiregular mappings, remains in doubt. Whether a bounded quasiregular mapping in $\mathbb{B}^n$ has at least one radial limit or not is an open question for $n \geq 3$. One could ask for the existence of radial limits under a different condition on the mappings. If the boundedness condition is replaced by asking that the mapping f satisfies a Dirichlet condition

$$\int_{B^n} |Df(x)|^p \, dx < \infty \tag{1.1}$$

then it is natural to expect that radial limits exist, except in a set of p-capacity zero. This is the case since the trace of a $W^{1,p}(\mathbb{B}^n)$ function on $\partial\mathbb{B}^n$ is defined up to a set of p-capacity zero. When u is a harmonic function and $p = 2$, this was proved by Beurling [B]. This theorem has been generalized in many directions. The function u does not have to be harmonic. It is enough for u to be continuous and in $W^{1,p}(\mathbb{B}^n)$ (see [C, section 5] for $n = 2$ and [M1] for the case $n \geq 3$). All these results involve radial limits. Indeed, they are not true if radial is replaced by nontangential (see [M2]). What is needed to pass from radial to nontangential limits is a normality condition that is implied by the weak Harnack inequality satisfied by solutions of elliptic pde. See [KMV] for details. For the special case of quasiregular mappings, Miklyukov in [M] proved that a bounded quasiregular mapping $f: \mathbb{B}^n \to \mathbb{R}^n$ satisfying (1.1) has nontangential limits everywhere on $\partial\mathbb{B}^n$ with the possible exception of a set of p-capacity zero. The boundedness hypothesis is used by Miklyukov to show that f is normal (uniformly continuous with respect to the hyperbolic metric in $\mathbb{B}^n$), which is always true for $p = n$.

The goal of this paper is to show that one can still replace radial by nontangential limits for the class of monotone functions in $W^{1,p}(\mathbb{B}^n)$, $n - 1 < p \leq n$. Our main result is:

Theorem 1. *Let $u: \mathbb{B}^n \to \mathbb{R}$ be a monotone function in the Sobolev space $W^{1,p}(\mathbb{B}^n)$ satisfying (1.1), where $n - 1 < p \leq n$. Let E be the set on the boundary of the unit ball where the nontangential limit does not exist, then E has p-capacity zero.*

The coordinate functions of quasiregular mappings, harmonic functions, as well as solutions to a wide range of elliptic partial differential equations, are monotone (as it can be easily deduced from the maximum principle). Therefore, our theorem implies that any continuous function $u: \mathbb{B}^n \to \mathbb{R}$ in the Sobolev class $W^{1,p}(\mathbb{B}^n)$, $n-1 < p < n$, satisfying weak maximum and minimum principles has nontangential limits at any point in $\partial\mathbb{B}^n$ except possibly in a set of p-capacity zero.

Given $u \in W^{1,p}(\mathbb{B}^n)$, its trace on $\partial\mathbb{B}^n$, denoted again by u, can be defined by extending u to $W^{1,p}(\mathbb{R}^n)$, approximating by smooth functions and using the Poincaré inequality

$$\int_{\partial\mathbb{B}^n} |u|^p \, dS \le c(n,p) \int_{\mathbb{B}^n} (|u|^p + |\nabla u|^p) \, dx. \tag{1.2}$$

We refer to [E, chapter 5] for details. Define $u_t(x) = u(tx)$ for $0 < t \le 1$ and $x \in \partial\mathbb{B}^n$. It turns out that $u_t \to u$ in $L^p(\partial\mathbb{B}^n)$ as $t \to 1^-$(see Lemma 3.3 below). Therefore, for a sequence $t_k \to 1^-$ we may recover the trace by taking radial limits

$$\lim_{t_k \to 1^-} u(t_k x) = u(x)$$

for a. e. $x \in \partial\mathbb{B}^n$. In fact, by using [M1], one can replace $t_k \to 1^-$ by $t \to 1^-$. Theorem 1 improves this convergence to the trace, when $n - 1 < p < n$ and u is monotone, replacing radial by nontangential limits.

Our proof is based on using the module method to establish the following extension of the classical Lindelöf theorem.

Theorem 2. *Let $u\colon \mathbb{B}^n \to \mathbb{R}$ be a monotone function in the class $W^{1,p}(\mathbb{B}^n)$, where $n - 1 < p \le n$. Then, for any $\epsilon > 0$, there exists an open set U in $\mathbb{R}^n$ such that $B_{1,p}(U) < \epsilon$ and for any $x_0 \in \partial\mathbb{B}^n \setminus U$ and γ any rectifiable curve ending at x_0 in $\mathbb{B}^n$ such that*

$$\lim_{x \to x_0,\ x \in \gamma} u(x) = \alpha,$$

then $u(x)$ has nontangential limit α at x_0. If $p = n$ the exceptional set U is empty.

Here $B_{1,p}(U)$ denotes the Bessel p-capacity of U. The case $p > n$ is vacuous since functions $W^{1,p}(\mathbb{B}^n)$ are continuous in $\bar{\mathbb{B}}^n$. Theorem 2 in the case $p = n$ is due to Vuorinen [V, chapter 4]. The thrust of our theorem is to get p below n. The limitation $p > n - 1$ appears in a module estimate (Lemma 2.2 below) on $(n-1)$-dimensional spheres.

The plan of the paper is as follows. In section §2 we present the relevant definitions and some lemmas needed later on. Section §3 contains some facts about Sobolev spaces that we could not find a reference for. In section §4, we present a key lemma showing roughly that, except for a set of small Bessel p-capacity, monotone functions in $W^{1,p}(\mathbb{B}^n)$, $n-1 < p < n$, are indeed uniformly continuous with respect to the hyperbolic metric in $\mathbb{B}^n$. Finally, we present the proofs of Theorems 1 and 2 in section §5.

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§2. Preliminaries.

The open ball centered at x_0 with radius r is denoted by $B^n(x_0,r)$. Its boundary is the $(n-1)$-dimensional sphere $S^{n-1}(x_0,r)$. By a cap of a sphere $S^{n-1}(x_0,r)$ we mean a set $H \cap S^{n-1}(x_0,r)$, where H is an open half space in $\mathbb{R}^n$. The spherical distance between two points in $\mathbb{R}^n$ is denoted by $q(x,y)$. It is used as a convenient device to avoid complications at infinity. For a point $x \in \partial\mathbb{B}^n$ we write $C(x)$ for the Stolz cone at x with a fixed given aperture. It follows that we can find a constant $c_n \geq 1$, depending only on the aperture and n, such that if $y \in C(x)$ then

$$|y-x| \leq c_n(1-|y|). \tag{2.1}$$

For an open set $G \subset \mathbb{R}^n$, the Sobolev space of functions $u \in L^p(G)$ such that the first distributional derivatives are also in $L^p(G)$ is denoted $W^{1,p}(G)$. It is endowed with the norm

$$\|u\|_{1,p} = \left(\int_G |u(y)|^p\,dy + \int_G |\nabla u(y)|^p\,dy\right)^{1/p}.$$

By $c(\alpha,\beta,\dots)$ we denote a constant that depends only on the parameters $\alpha,\beta,\dots$ and that may change value from line to line.

Let be a family of curves in $\mathbb{R}^n$. Denote by $\mathcal{F}(\Gamma)$ the collection of admissible metrics for Γ, i.e. nonnegative Borel measurable functions $\rho\colon \mathbb{R}^n \to \mathbb{R}\cup\{\infty\}$ such that

$$\int_\gamma \rho\,ds \geq 1$$

for each locally rectifiable curve $\gamma \in \Gamma$. For $p \geq 1$ the p-module of Γ is defined by

$$M_p(\Gamma) = \inf_{\rho\in\mathcal{F}(\Gamma)} \int_{\mathbb{R}^n} \rho^p\,dx.$$

If $\mathcal{F}(\Gamma) = \emptyset$, we set $M_p(\Gamma) = \infty$.

The same definition applies to families of curves that lie in a $(n-1)$-dimensional submanifold S of $\mathbb{R}^n$, replacing the Lebesgue measure by the corresponding surface measure. In this case, we use the notation $M_p^S(\Gamma)$.

The following upper bound for modules can be found in [V, lemma 7.6].

Lemma 2.1. *Let G be a domain in $\mathbb{R}^n$ and $u: G \to \mathbb{R}$ be a continuous function in $W^{1,p}(G)$, where $p > 1$. Let $-\infty < a < b < \infty$, and A, B be nonempty subsets of a ball $B^n(x_0,r) \subset G$ such that $u(x) \leq a$ for any $x \in A$ and $u(x) \geq b$ for any $x \in B$. Then, we have*

$$M_p(\Delta(A,B;B^n(x_0,r))) \leq \frac{1}{(b-a)^p}\int_{B^n(x_0,r)} |\nabla u(x)|^p\,dx,$$

where $\Delta(A,B;B^n(x_0,r))$ is the family of all rectifiable curves in $B^n(x_0,r)$ joining A and B.

Lower bounds for modules are naturally harder to find. The next lemma is an adaptation of [Va, theorem 10.2] to the case $n-1 < p \leq n$.

Lemma 2.2. *Let $n \geq 2$, and K be a cap of the sphere $S^{n-1}(x_0, r)$. Suppose that E and F are disjoint nonempty sets of $\bar{K}$. Let $\Delta(E, F; K)$, be the family of all rectifiable curves in K joining E and F, then*

$$M_p^{S^{n-1}(x_0,r)}(\Delta(E, F; K)) \geq \frac{c(n,p)}{r^{p-n+1}}$$

where $n-1 < p \leq n$, and $c(n,p)$ is a constant depending only on n and p.

The proof is exactly the same as in the case $p = n$ given in [Va, page 29] using Hölder's inequality with exponent p instead of n and noting that the integrals appearing are finite for $n - 1 < p \leq n$.

Let $E \subset \mathbb{R}^n$. The Bessel p-capacity of E is denoted by $B_{1,p}(E)$. We refer to the book [Z] for the definition and properties of $B_{1,p}$. The family of all rectifiable curves in $\mathbb{R}^n$ that intersect E is denoted by $\aleph(E)$. The p-dimensional module of E is the quantity $M_p(\aleph(E))$. We shall use the following version of a theorem proved by Ziemer [Z2]: *If E is compact, then it has p-dimensional module zero if and only if it has Bessel p-capacity zero.*

Next, we recall the definition of monotone function.

Definition. *Let $G \subset \mathbb{R}^n$ be an open set. A continuous function $u: G \to \mathbb{R}$ is said to be monotone (in the sense of Lebesgue) if*

$$\max_{\bar{D}} u(x) = \max_{\partial D} u(x)$$

and

$$\min_{\bar{D}} u(x) = \min_{\partial D} u(x)$$

hold whenever D is a domain with compact closure $\bar{D} \subset G$.

It follows from the above definition that if $t \in \mathbb{R}$, then each component $A \neq \emptyset$ of the set $\{z \in G: u(z) > t\}$ fails to be relatively compact, i.e. $\bar{A} \cap \partial G \neq \emptyset$. A similar statement holds if $>$ is replaced by $\geq$, $<$ or $\leq$.

§3. Pointwise behavior of Sobolev functions

Lemma 3.1. *Let $u \in W^{1,p}(\mathbb{R}^n)$ where $1 < p < n$. Then*

$$B_{1,p}\left(\left\{x \in \mathbb{R}^n : \sup_{r>0}\left[r^p \fint_{B^n(x,r)} |\nabla u(y)|^p \, dy\right]^{1/p} > t\right\}\right) \le \frac{c(p,n)}{t^p}\|u\|_{1,p}^p.$$

Proof. Fix $s > 0$ and set

$$E_t = \left\{x \in \mathbb{R}^n : \sup_{r>0}\left[r^{-s}\int_{B^n(x,r)} |\nabla u(y)|^p \, dy\right]^{1/p} > t\right\}$$

Since $\nabla u \in L^p(\mathbb{R}^n)$ a standard estimate ([HKM, lemma 2.30) gives

$$\Lambda_s^\infty(E_t) \le \frac{c(s)}{t^p}\int_{\mathbb{R}^n} |\nabla u(y)|^p \, dy, \tag{3.1}$$

where

$$\Lambda_s^\infty(E) = \inf\left\{\sum r_i^s : E \subset \bigcup B^n(z_i, r_i)\right\}$$

is the s-Hausdorff content of E. Choose a cover of E_t such that $\sum r_i^s \le 2\Lambda_s^\infty(E_t)$. Taking $s = n - p > 0$ we have

$$B_{1,p}(E_t) \le \sum_i B_{1,p}(B^n(z_i, r_i)) \le c(p,n)\sum_i r_i^{n-p}, \tag{3.2}$$

where we have used the fact that the Bessel p-capacity of $B^n(z,r)$ is r^{n-p} except for constants. From (3.2) we deduce

$$B_{1,p}(E_t) \le c(p,n)\Lambda_{n-p}^\infty(E_t),$$

and we finish by using (3.1). □

Lemma 3.2. *Let $u \in W^{1,p}(\mathbb{R}^n)$ where $1 < p < n$. Then for every $\epsilon > 0$ there exists an open set $U \subset \mathbb{R}^n$ with $B_{1,p}(U) < \epsilon$ such that*

$$\lim_{r \to 0} r^p \fint_{B(x,r)} |\nabla u(y)|^p \, dy = 0$$

uniformly on $\mathbb{R}^n \setminus U$.

Proof. For $x \in \mathbb{R}^n$ and $r > 0$, set

$$A_r u(x) = r^p \fint_{B(x,r)} |\nabla u(y)|^p \, dy.$$

Fix a small $\eta \in (0,1)$, choose $g_\eta \in C_0^\infty(\mathbb{R}^n)$ such that $\|u - g_\eta\|_{1,p}^p \leq \eta^2$ and put $h_\eta = u - g_\eta$. Then we write

$$A_r u(x) \leq c(p)\big[A_r g_\eta(x) + A_r h_\eta(x)\big].$$

Choose $r_\eta > 0$ such that

$$\sup_{0<r<r_\eta} [c(p) A_r g_\eta(x)] < \eta$$

whenever $x \in \mathbb{R}^n$. We have

$$\{x \in \mathbb{R}^n\colon \sup_{0<r<r_\eta} A_r u(x) > 2\eta\} \subset \{x \in \mathbb{R}^n\colon \sup_{0<r<r_\eta} A_r h_\eta(x) > 2\frac{\eta}{c(p)}\}.$$

By the preceding lemma,

$$B_{1,p}\left[\{x \in \mathbb{R}^n\colon \sup_{0<r<r_\eta} A_r u(x) > 2\eta\}\right] \leq \frac{c(p,n)}{\eta} \|h_\eta\|_{1,p}^p \leq c(p,n)\eta.$$

For a sequence of positive numbers $\delta_i \to 0$ we define

$$U = \bigcup_{i=1}^\infty \{x \in \mathbb{R}^n\colon \sup_{0<r<r_{\delta_i}} A_r u(x) > \delta_i\}.$$

The set U is open and

$$B_{1,p}(U) \leq c(p,n) \sum \delta_i.$$

Choosing δ_i small enough so that $c(p,n)\sum \delta_i \leq \epsilon$ we see that if $x \notin U$, $A_r u(x) \leq \delta_i$ for $r \leq r_{\delta_i}$. □

Lemma 3.3. *Let $u \in W^{1,p}(\mathbb{B}^n)$, where $1 \leq p < \infty$. Then $u_t(x) = u(tx)$ converges to the trace of u on $L^p(\partial\mathbb{B}^n)$ as $t \to 1^-$.*

Proof. Select a sequence u_m of $C^\infty(\bar{\mathbb{B}}^n)$ tending to u in $W^{1,p}(\mathbb{B}^n)$. Write

$$u(tx) - u(x) = u(tx) - u_m(tx) + u_m(tx) - u_m(x) + u_m(x) - u(x).$$

Use the Poincaré inequality (1.2) twice to obtain

$$\|u_t - u\|_p \leq c(n,p)\left(\|u - u_m\|_{1,p} + \left(\int_{S^{n-1}} |u_m(tx) - u_m(x)|^p\right)^{1/p}\right).$$

Given $\epsilon > 0$, choose m so large that $c(n,p)\|u - u_m\|_{1,p} \leq \epsilon/2$. Then choose t_0 so that for $t \in (t_0, 1)$

$$\left(\int_{S^{n-1}} |u_m(tx) - u_m(x)|^p\right)^{1/p} \leq \epsilon/2,$$

to conclude that $\|u_t - u\|_p < \epsilon$ for $t \in (t_0, 1)$. □

§4. Boundary behavior of monotone Sobolev functions

Lemma 4.1. *Let $u: \mathbb{B}^n \to \mathbb{R}$ be a monotone function in the Sobolev class $W^{1,p}(\mathbb{B}^n)$, where $n-1 < p < n$. Then for any $\epsilon > 0$, there exists an open set U in $\mathbb{R}^n$ satisfying $B_{1,p}(U) < \epsilon$, such that: if $x_0 \in \partial\mathbb{B}^n \setminus U$ and $\{b_k\}_{k=1}^{\infty}$ is a sequence contained in the Stolz cone $C(x_0)$ such that $\lim_{k\to\infty} b_k = x_0$ and $\lim_{k\to\infty} u(b_k) = \beta$, then for any $\eta > 0$ there exist an integer $k_0 \geq 1$ such that*

$$q(u(x), \beta) < \eta$$

for any $x \in E = \bigcup_{k \geq k_0} B^n(b_k, \frac{1}{4}(1-|b_k|))$.

Proof. We extend u to a function in $W^{1,p}(\mathbb{R}^n)$ with comparable norm and denote it again by u. Fix $\epsilon > 0$ and choose U given by Lemma 3.2. For a constant δ_0 to be determined later, choose $r_0 > 0$ such that for $0 < r < r_0$ and $x_0 \in \mathbb{R}^n \setminus U$,

$$\int_{B^n(x_0,r)} |\nabla u(y)|^p \, dy \leq \delta_0 r^{n-p}. \tag{4.1}$$

Select a point $y \in B^n(b_k, \frac{1-|b_k|}{4})$. We may assume that $u(b_k) < u(y)$ (the case $u(b_k) > u(y)$ is handled by a symmetric argument). Set

$$A = \{z \in B^n(b_k, \frac{1-|b_k|}{2}) : u(z) \leq u(b_k)\}$$

and

$$B = \{z \in B^n(b_k, \frac{1-|b_k|}{2}) : u(z) \geq u(y)\}.$$

Since u is monotone we know that

$$A \cap S^{n-1}(b_k, t) \neq \emptyset$$

and

$$B \cap S^{n-1}(b_k, t) \neq \emptyset$$

for $|b_k - y| < t \leq \frac{1-|b_k|}{2}$. Use now Lemma 2.2 with $K = S^{n-1}(b_k, t)$ to obtain

$$M_p^{S^{n-1}(b_k,t)}(\Delta(A \cap S^{n-1}(b_k,t), B \cap S^{n-1}(b_k,t); S^{n-1}(b_k,t))) \geq \frac{c(n,p)}{t^{p-n+1}}.$$

Integrating in t we get

$$M_p(\Delta(A, B, B^n(b_k, \frac{1-|b_k|}{2}) \setminus B^n(b_k, |b_k - y|))) \geq c(n,p) \int_{|b_k-y|}^{\frac{1-|b_k|}{2}} \frac{1}{t^{p-n+1}} \, dt.$$

By monotonicity of the module we obtain

$$(4.2) \qquad M_p(\Delta(A, B, B^n(b_k, \frac{1-|b_k|}{2}))) \geq c(n,p)(1-|b_k|)^{n-p}.$$

Lemma 2.1 gives now

$$\frac{1}{|u(b_k)-u(y)|^p} \int_{B^n(b_k, \frac{1-|b_k|}{2})} |\nabla u(x)|^p \, dx \geq c(n,p)(1-|b_k|)^{n-p}.$$

Since $b_k \in C(x_0)$ one easily checks using (2.1) that

$$B^n(b_k, \frac{1-|b_k|}{2}) \subset B^n(x_0, d_n(1-|b_k|)),$$

where $d_n = c_n + 1/2$. Therefore

$$\int_{B^n(x_0, d_n(1-|b_k|))} |\nabla u(x)|^p \, dx \geq c(n,p)|u(b_k)-u(y)|^p (1-|b_k|)^{n-p}.$$

Choose k_1 so that for $k > k_1$ we have $d_n(1-|b_k|) < r_0$. We obtain

$$|u(b_k)-u(y)|^p \leq c(n,p)\delta_0 d_n^{p-n}$$

for $y \in B^n(b_k, \frac{1}{4}(1-|b_k|))$.

Given any positive η choose $k_0 > k_1$ such that $|u(b_k)-\beta| < \eta/2$ for $k > k_0$. Choose δ_0 such that $(c(n,p)\delta_0 d_n^{p-n})^{1/p} < \eta/2$. Then, for any $x \in E$ we have

$$|u(x)-\beta| \leq |u(x)-u(b_{k'})| + |u(b_{k'})-\beta| < \eta$$

for some $k' \geq k_0$. $\square$

Remark. Indeed, the above proof gives the following oscillation estimate for monotone Sobolev functions in $W^{1,p}(\Omega)$, where Ω is a domain in $\mathbb{R}^n$ and $n-1 < p < n$:

$$(4.3) \qquad (\operatorname{osc}(u, B^n(x,r)))^p \leq c(n,p) r^p \fint_{B^n(x,2r)} |\nabla u(y)|^p \, dy,$$

whenever $B^n(x,2r) \subset \Omega$. The analogue of (4.3) in the case $p = n$ is well known [Va, chapter 4] and it takes the form

$$(\operatorname{osc}(u, B^n(x,r)))^n \leq c(n) \frac{1}{\log(R/r)} \int_{B^n(x,R)} |\nabla u(y)|^n \, dy,$$

whenever $B^n(x,R) \subset \Omega$.

§5. Proof of Theorems 1 and 2

Proof of Theorem 2 for $n-1<p<n$. Given $\epsilon>0$ choose U given by Lemma 4.1. Note that after extending u to $\mathbb{R}^n$ as in the proof of Lemma 4.1 the conclusion of Lemma 3.2 applies in $\mathbb{R}^n \setminus U$. Without lost of generality we may assume that $x_0 = \mathbf{1} = (1,0,\dots,0)$. The proof proceeds by contradiction. Suppose that we have a sequence $\{b_k\}$ in a Stolz cone $C(\mathbf{1})$ such that $\lim_{k\to\infty} b_k = \mathbf{1}$ and

$$\lim_{k\to\infty} u(b_k) = \beta \neq \alpha.$$

Assume that $-\infty < |\alpha| < |\beta| < \infty$ (the other cases being similar or easier). Let $\eta = (|\beta|-|\alpha|)/6$ and choose k_1 such that for $k>k_1$ we have

$$|u(x)| < |\alpha| + \eta \text{ for } x \in \gamma \cap B^n(\mathbf{1}, |1-b_k|)$$

and

$$|u(b_k)| > |\beta| - \eta.$$

Choose k_2 given by Lemma 4.1 so that for $k > k_0 = \max(k_1,k_2)$

$$|u(x)| > |\beta| - 2\eta, \text{ for } x \in B^n(b_k, \frac{1}{4}(1-|b_k|)).$$

Consider now the following module problem. Denote by

$$H_k = B^n \cap [B^n(\mathbf{1}, |1-b_k|) \setminus \bar{B}^n(\mathbf{1}, |1-b_k| - \frac{1}{8}(1-|b_k|))]$$

and let

$$E = \gamma \cap H_k$$

and $F = B^n(b_k, \frac{1}{4}(1-|b_k|)) \cap H_k$. We will estimate the quantity $M_p(\Delta(E,F;H_k))$. To get un upper bound, observe that for any locally rectifiable curve l joining E and F in H_k we have

$$\int_l |\nabla u(x)|\, ds \geq \frac{|\beta|-|\alpha|}{2}.$$

Thus, the metric $\rho = \frac{2}{|\beta|-|\alpha|}|\nabla u|$ is admissible. From the definition of module we obtain

$$M_p(\Delta(E,F;H_k)) \leq \frac{c(p)}{(|\beta|-|\alpha|)^p} \int_{H_k} |\nabla u(x)|^p\, dx. \tag{5.1}$$

On the other hand, considering the spherical caps

$$K_t = \mathbb{B}^n \cap S^{n-1}(1,t) \text{ for } t \in (|1-b_k| - \frac{1}{8}(1-|b_k|), |1-b_k|),$$

and the nonempty disjoint sets in K_t, $E_t = S^{n-1}(1,t)\cap E$ and $F_t = S^{n-1}(1,t)\cap F$, we have that

$$M_p^{S^{n-1}(1,t)}(\Delta(E_t, F_t; K_t)) \geq \frac{c(n,p)}{t^{p-n+1}}.$$

Integrating in t we get

$$M_p(\Delta(E, F; H_k)) \geq \int_{|1-b_k|-\frac{1}{8}(1-|b_k|)}^{|1-b_k|} \frac{c(n,p)}{t^{p-n+1}}\, dt.$$

Since the sequence $\{b_k\}$ is in the Stolz cone $C(\mathbf{1})$ we have by (2.1)

$$M_p(\Delta(E, F; H_k)) \geq c(n,p)(|1-b_k|)^{n-p}. \tag{5.2}$$

Combining (5.1) and (5.2) we have

$$\begin{aligned} c(n,p)(|1-b_k|)^{n-p} &\leq \frac{c(p)}{(|\beta|-|\alpha|)^p}\int_{H_k} |\nabla u(x)|^p\, dx \\ &\leq \frac{c(p)}{(|\beta|-|\alpha|)^p}\int_{B^n(\mathbf{1},|1-b_k|)} |\nabla u(x)|^p\, dx. \end{aligned}$$

Therefore we have the following inequality

$$0 < c(n,p) \leq \frac{c(n,p)}{(|\beta|-|\alpha|)^p}(|1-b_k|)^p \fint_{B^n(\mathbf{1},|1-b_k|)} |\nabla u(x)|^p\, dx.$$

Since, by assumption, $\mathbf{1}\notin U$, the right hand side of the above inequality goes to zero as k goes to ∞, which gives a contradiction. $\square$

Proof of Theorem 1. The exceptional set E is a Borel set as an elementary argument shows. Thus, to show that E has p-capacity zero it is enough to show that any compact subset K of E has p-capacity zero. Fix such a K. For any arbitrary $\epsilon > 0$, let U be an open set in $\mathbb{R}^n$ given by Theorem 2. Let γ be any rectifiable curve in $\mathbb{B}^n$ ending at a point in $(\partial\mathbb{B}^n\setminus U)\cap K$.

Since the nontangential limit does not exist at any point in K, Theorem 2 implies that the limit through γ does not exist either. Therefore

$$\int_\gamma |\nabla u(x)|\, ds = \infty. \tag{5.3}$$

Let $\Gamma((\partial\mathbb{B}^n\setminus U)\cap K)$ be the family of all the rectifiable curves in $\mathbb{B}^n$ that intersect $(\partial\mathbb{B}^n\setminus U)\cap K$.

Our goal is to show that the module $M_p(\Gamma((\partial\mathbb{B}^n\setminus U)\cap K))$ is zero. For this, we choose the metric in $\mathbb{B}^n$ given by $\rho_\eta(x) = \eta|\nabla u(x)|$. It follows from (5.3) that for any positive η, the metric ρ_η is admissible for our module problem. Hence we have

$$M_p(\Gamma((\partial\mathbb{B}^n \setminus U) \cap K)) \leq \eta^p \int_{\mathbb{B}^n} |\nabla u(x)|^p \, dx < \eta^p \|u\|_{1,p}^p.$$

Letting η go to 0 we obtain that $M_p(\Gamma((\partial\mathbb{B}^n \setminus U) \cap K)) = 0$. Next, we use a symmetrization lemma [V, Lemma 5.22] for modules to obtain

$$M_p(\Gamma((\partial\mathbb{B}^n \setminus U) \cap K)) \geq \frac{1}{2} M_p(\aleph((\partial\mathbb{B}^n \setminus U) \cap K)).$$

Thus, we conclude that $M_p(\aleph((\partial\mathbb{B}^n \setminus U) \cap K)) = 0$. By the theorem of Ziemer quoted in the preliminaries, this implies that $B_{1,p}((\partial\mathbb{B}^n \setminus U) \cap K) = 0$. Finally, by subadditivity of the Bessel capacities we have

$$B_{1,p}(K) \leq B_{1,p}((\partial\mathbb{B}^n \setminus U) \cap K) + B_{1,p}(U) < \epsilon.$$

Since ϵ is independent of K, letting ϵ go to 0 we obtain that $B_{1,p}(K) = 0$. □

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