5.5 The real zeros of a polynomial function

Recall, that c is a zero of a polynomial f(x), if f(c) = 0.

Example:

a) Find real zeros of f(x)= $x^2 + x - 1$. We need to find x for which f(x) = 0, that is we have to solve the equation $x^2 + x - 1 = 0$. Using quadratic formula we get $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 - 4(1)(-1)}}{2} = \frac{-1 \pm \sqrt{5}}{2}$

The equation has two solutions , therefore f(x)= $x^2 + x - 1$ has two zeros $\frac{-1-\sqrt{5}}{2}$, $\frac{-1+\sqrt{5}}{2}$

b) Find real zeros of f(x) = $x^2 - 3$. To find zeros we need to solve the equation $x^2 - 3 = 0$. Using the square root method we get $x^2 = 3$ and $x = \pm\sqrt{3}$. Hence the zeros are $-\sqrt{3}$ and $+\sqrt{3}$.

c) Find real zeros of f(x) = $x^2 + 4$. The equation $x^2 + 4 = 0$ has no real solution ($x^2 = -4$ is never true, no matter what x is). Therefore f(x) = $x^2 + 4$ has no real zeros.

Theorem ***Division Algorithm for Polynomials***

Suppose that f(x) and g(x) are polynomials and that the degree of g is not zero (g(x) is not a constant). Then there are unique polynomials Q(x) and R(x) such that

$$\frac{f(x)}{g(x)} = Q(x) + \frac{R(x)}{g(x)} \quad or \quad f(x) = Q(x) \cdot g(x) + R(x)$$

where the degree of R(x) < the degree of g(x).

f(x) is called the **dividend**, g(x) is called the **divisor**, Q(x) is the **quotient** and R(x) is the **remainder.**

If g(x) = (x-c) , then f(x) = (x-c)Q(x) + R(x) , and the degree of R(x) < the degree of g(x)= 1. Hence, R(x) is a constant, R(x) = R. Note also that when x = c, we get f(c) = (c-c)q(c) + R = R.

Theorem ***Remainder Theorem***

If f is a polynomial, then when f(x) is divided by (x-c), the remainder R = f(c).

Factor Theorem

(x-c) is a factor of a polynomial f(x) if and only if f(c) = 0

Proof: f(x) = (x-c)Q(x) + f(c)

Example:

a) Let f(x) = $4x^4 - 15x^2 - 4$. Find the remainder when f(x) is divided by x-2

When f (x) is divided by (x-2), then the remainder R = f(2) = $4(2)^4-15(2)^2- 4$= 64 -60-4 = 0. Hence, (x-2) is a factor of f(x), that is f(x) = (x-2)Q(x), where Q(x) is a polynomial.

b) Let f(x) = $2x^6 - 18x^4 + x^2 - 9$. Find the remainder when f(x) is divided by x+ 1.

Note first that x+1 = x-(-1).

When f (x) is divided by (x –(- 1)), then the remainder is

$$R = f(-1) = 2(-1)^6 - 18(-1)^4 + (-1)^2 - 9 = 2-18+1-9 = -24$$

Therefore (x+1) is not a factor of f(x).

Theorem

A polynomial of degree n can have at most n real zeros.

Example: How many real zeros can the polynomial $f(x) = 2x^6 - 18x^4 + x^2 - 9$ have?

Degree of f(x) = 6, so there are at most 6 real zeros.

Example:

a) Since $f(x) = x^2 + x - 1$ has two real zeros $\frac{-1-\sqrt{5}}{2}$ and $\frac{-1+\sqrt{5}}{2}$, then $\left(x - \frac{-1-\sqrt{5}}{2}\right)$ and $\left(x - \frac{-1+\sqrt{5}}{2}\right)$ are factors of f(x), and $f(x) = x^2 + x - 1 = \left(x - \frac{-1-\sqrt{5}}{2}\right)\left(x - \frac{-1+\sqrt{5}}{2}\right)$

b) Similarly, $f(x) = x^2 - 3 = \left(x - \sqrt{3}\right)\left(x - (-\sqrt{3})\right) = (x - \sqrt{3})(x + \sqrt{3})$

c) $f(x) = x^2 + 4$ has no real zeros, so it cannot be factored over the real numbers. We call such a polynomial **irreducible** or **prime**

Theorem

Every polynomial function with real coefficients can be uniquely factored into a product of linear factors (x-c) and /or irreducible quadratic factors $(ax^2 + bx+c)$

Theorem

A polynomial function with real coefficients and with odd degree must have at least one real zero.

Theorem ***Rational Zeros Theorem***

Supose $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, $a_n, a_0 \neq 0$, where all coefficients a_k are integers, is a polynomial of degree greater than 0. If p/q is a rational zero of f(x), then p divides a_0 and q divides a_n.

In other words, any rational zero of a polynomial with integer coefficients, must be of the form

$$\frac{factor\ of\ a_0}{factor\ of\ a_n}$$

Example:

List all potential zeros of the polynomial $f(x) = -4x^3 + x^2 + x + 6$

$a_3 = -4$; factors: ±1, ±2, ±4 (p)

$a_0 = 6$; factors : ±1, ±2, ±3 , ±6 (q)

$$\frac{p}{q} = \frac{factor\ of\ a_0}{factor\ of\ a_n} = \pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{1}{4}, \pm \frac{3}{4}$$

Remark: A polynomial function with integer coefficients might not have any rational (or even real) zeros. For example $f(x) = x^2 - 2$ has no rational zeros (the zeros are $\pm\sqrt{2}$), and $f(x) = x^2 + 2$ has no real zeros at all.

Once we know potential zeros, we can check whether they are actual zeros, either by computing value of the polynomial at the potential zero or performing division.

Example:

Find all real zeros of the given polynomial and then factor the polynomial

$f(x) = x^3 + 8x^2 + 11x - 20$

- degree of f = 3; so there are at most 3 zeros
- all coefficients are integers, so we will look for rational zeros among the numbers of the form $\frac{factor\ of\ a_0}{factor\ of\ a_n}$

  $a_0 = -20$; factors of a_0 : ±1, ±2, ±4 , ±5, ±10, ±20
  $a_n = 1$; factors of a_n : ±1
  $$\frac{p}{q} = \frac{factor\ of\ a_0}{factor\ of\ a_n} = \pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20$$

(i) Is x= 1 a zero?

$f(1)= (1)^3 + 8(1)^2 + 11(1) -20= 1+8+11-20 = 0$; So, **x = 1 is a zero**

Therefore, $f(x) = x^3 + 8x^2 + 11x - 20 = (x-1)Q(x)$.

To find Q(x), we must divide $f(x) = x^3 + 8x^2 + 11x - 20$ by (x-1). We can use either **long division** or a simplified version called **synthetic division** (applied ONLY when the divisor is of the form (x-c))

Long division:

$x^2 + 9x + 20$ *this is the **quotient Q(x)***

$x - 1 \,)\, x^3 + 8x^2 + 11x - 20$ divide x^3 by x

$\underline{x^3 - x^2}$ multiply $x^2(x-1)$ and subtract;

$9x^2 + 11x - 20$ divide $9x^2$ by x

$\underline{9x^2 - 9x}$ multiply $9x(x-1)$ and subtract;

$20x - 20$ divide 20x by x

$\underline{20x - 20}$ multiply 20(x-1) and subtract

0 *this is the **remainder R***

Synthetic division:

1	1	8	11	−20
		1	9	20
	1	9	20	0
	coefficients of the quotient			remainder

So , **Q(x) = x^2 + 9x + 20** and f(x) = $x^3 + 8x^2 + 11x - 20 = (x-1)(x^2 + 9x+20)$.

Now we must find zeros of Q(x) = $x^2 + 9x + 20$. Since it is a quadratic polynomial we solve the equation Q(x) = 0 or $x^2 + 9x + 20 = 0$ either by factoring or using the quadratic formula (if factoring does not work).

Since $x^2 + 9x + 20 = (x+4)(x+5)$, the equation becomes (x+4)(x+5) = 0
the zeros are -4 and -5.

Therefore the zeros of f(x) are 1, -4, -5 and f(x) = (x-1)(x+4)(x+5)

Solving polynomial equations

If f(x) is a polynomial, then the equation f(x) = 0 is called a polynomial equation. To solve such an equation, means to find zeros of the polynomial function f(x).

Example:

Solve the equation $2x^3 - 11x^2 + 10x + 8 = 0$

Let f(x) = $2x^3 - 11x^2 + 10x + 8$. We look for the rational zeros, since the equation has integer coefficients.

Factors of a_o are : ±1, ±2, ±4 , ±8
Factors of $a_n = a_3$ are : ±1, ±2
Possible zeros: $\frac{p}{q} = \frac{factor\ of\ a_0}{factor\ of\ a_n} = \pm 1, \pm 2, \pm 4, \pm 8, \pm \frac{1}{2}$

Is x= 1 a zero? f(1) = 2-11+10+8 = $9 \neq 0$, so x = 1 is **not** a zero

Is x = -1 a zero? f(-1) = -2 -11 – 10 + 8 = - 15 $\neq$ 0, so x = -1 is **not** a zero

Is x = 2 a zero? f(2) = 16 – 44+ 20 + 8 = 0, so x = 2 **is a zero**

Since x = 2 is a zero, then (x-2) is a factor of f(x), that is f(x) = (x-2)Q(x). To find Q(x) we'll use synthetic division

$$\begin{array}{r|rrrr} 2 & 2 & -11 & 10 & 8 \\ & & 4 & -14 & -8 \\ \hline & 2 & -7 & -4 & 0 \end{array}$$

And therefore, Q(x) = $2x^2 - 7x - 4$. Since Q(x) is a quadratic polynomial, we use the learned techniques to try to factor it. Indeed Q(x) = $2x^2 - 7x - 4 = (2x+1)(x-4)$.

Therefore f(x) = $2x^3 - 11x^2 + 10x + 8 = (x-2)(2x^2 - 7x - 4) = (x-2)(2x+1)(x - 4)$.

So the zeros, or the solutions of f(x) = 0, are 2, -1/2, 4.

5.1 Polynomial Functions

A **polynomial functions** is a function of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$

Example: $f(x) = 3x^3 – 2x^2 + 5x - 4$

The **domain** of a polynomial function is the set of all real numbers.

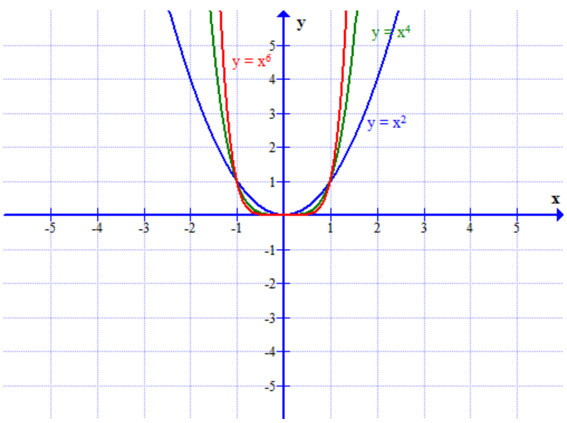
The **x-intercepts** are the solutions of the equation $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0$

The y-intercept is $y = f(0) = a_0$.

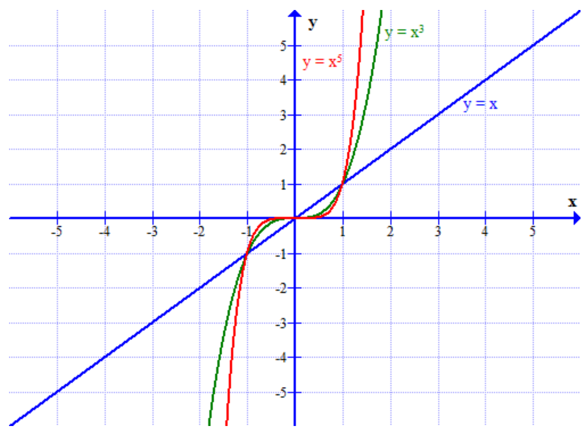
The graph of a polynomial function does not have holes or gaps (we say that a polynomial function is continuous) and it does not have sharp corners or cusps (we say it is smooth).

Power functions: $f(x) = x^n$, n is a natural number

The graphs of some power functions are given below



n- even

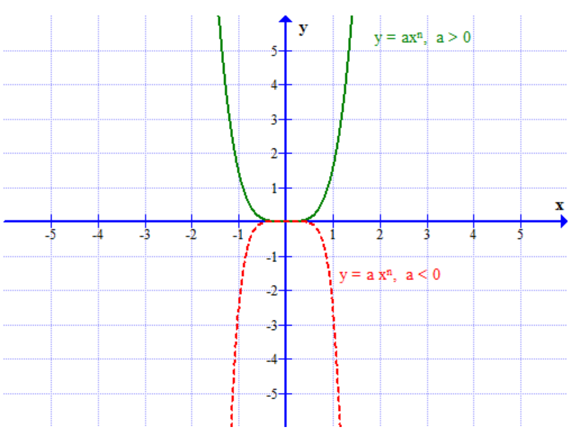


n- odd

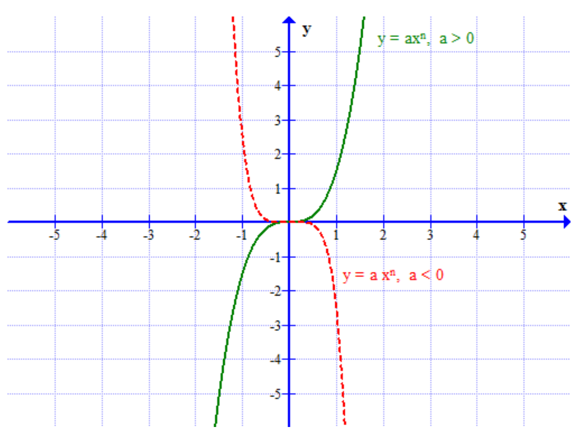
Notice that when n is even, a power function $y = x^n$ behaves like a parabola (graph is symmetric about the y-axis and contains points (-1,1), (0,0), (1,1)). When n is odd, a power function $y = x^n$, $n > 1$ has the graph similar to the cube function (symmetric about the origin, contains the points (-1,-1), (0,0), (1,1)).

Power function $f(x) = ax^n$, $a \neq 0$

The graph of $f(x) = ax^n$ is obtained from the graph of $y = x^n$ by stretching by a factor of a, if a is positive, and stretching by the factor of |a| and reflecting about the x-axis, if a is negative.



n- even



n – odd

Zeros of a polynomial function

If r is such a number that f(r) = 0, then r is called a **zero of the function** f.

If r is a zero of a polynomial function f, then we have the following:

(i) f(r) = 0

(ii) (r, 0) is an x-intercept of f

(iii) (x-r) is a factor of f(x), that is f(x) = (x-r)·q(x) (q(x) is the quotient in the division f(x) ÷ (x-r))

We say that r is a zero of **multiplicity** n, if n is the largest power, such that $f(x) = (x-r)^n\, q(x)$

Example: Let $f(x) = x^4 - 2x^3 + 3x^2 - x - 1$. Note that $f(1) = (1)^4 - 2(1)^3 + 3(1)^2 - (1) - 1 = 1-2+3-1-1 = 0$.

Therefore, r = 1 is a zero of f. This also means, that (1,0) is an x-intercept and that (x-1) is a factor of f(x), that is, when f(x) is divided by (x-1) the remainder is 0.

To find the other factor, q, we perform the division

$$
\begin{array}{rl}
 & \underline{x^3 - x^2 + 2x + 1} \\
x-1 \,\big) & x^4 - 2x^3 + 3x^2 - x - 1 \\
- & \underline{x^4 - x^3} \\
 & -x^3 + 3x^2 - x - 1 \\
- & \underline{-x^3 + x^2} \\
 & 2x^2 - x - 1 \\
- & \underline{2x^2 - 2x} \\
 & x - 1 \\
- & \underline{x - 1} \\
 & 0
\end{array}
$$

Therefore, $f(x) = x^4 - 2x^3 + 3x^2 - x - 1 = (x-1)(x^3 - x^2 + 2x + 1)$.

What is the multiplicity of that zero?

Since (x-1) is a factor of f(x), then the multiplicity of x = 1 is at least 1. If x= 1 had the multiplicity two, then f(x) would have $(x-1)^2$ as a factor. Which means that the quotient q above, $q(x) = x^3 - x^2 + 2x + 1$, would have (x-1) as a factor. If (x-1) were a factor of q(x), then x =1 would be a zero of q. But $q(1) = (1)^3 - (1)^2 + 2(1) + 1 = 3 \neq 0$. This means that (x-1) is not a factor of q(x) and consequently, $(x-1)^2$ is not a factor of f(x). Hence, the multiplicity of x = 1 is one.

Example: Find all zeros of function $f(x) = 2(x-3)(x+2)(x+3)^2(x-5)^4$ and determine their multiplicity

To find zeros, solve the equation f(x) = 0

$2(x-3)(x+2)(x+3)^2(x-5)^4 = 0$ (use the Zero Product property)

x-3 = 0 or x+2 = 0 or x+3= 0 or x- 5 = 0

So the zeros are : 3, -2, -3, 5

To find the multiplicities:

(i) factor the polynomial completely; use exponents to indicate multiple factors

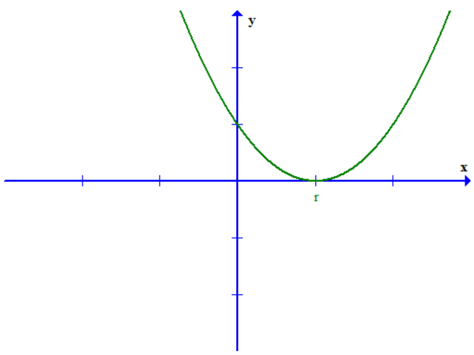
(ii) The multiplicity of a zero r is the exponent of the factor (x-r) that appears in the product

$$f(x) = 2(x-3)^{1}(x+2)^{1}(x+3)^{2}(x-5)^{4}$$

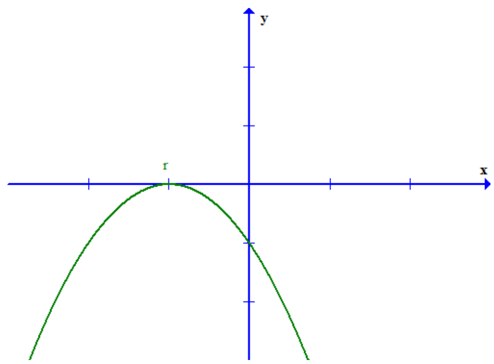
zeros	3	-2	-3	5
multiplicity	1	1	2	4

Suppose r is the zero of a polynomial function. (Remember that (r,0) is then an x-intercept)

If r is **a zero of even multiplicity**, then the graph of f will touch the x axis at the intercept (r, 0) as shown below.

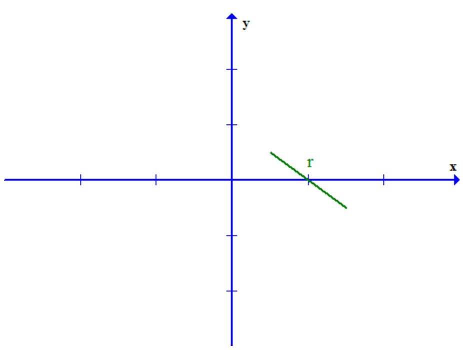


or



If r is a **zero of odd multiplicity**, then the graph of f will cross the x-axis at the intercept (r,0) as shown below.

Multiplicity one



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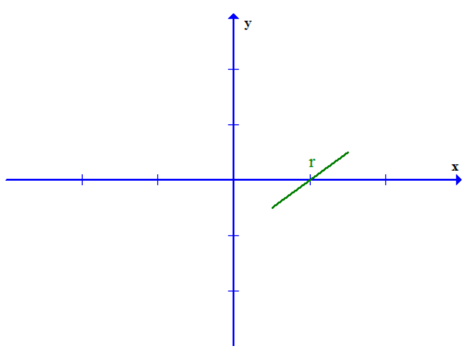
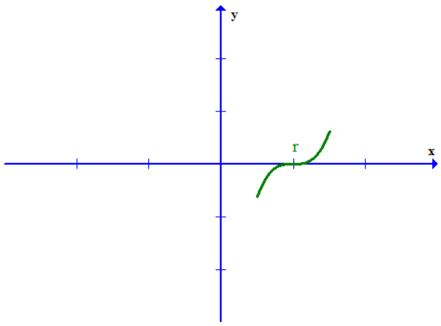


Multiplicity larger than 1 (odd)



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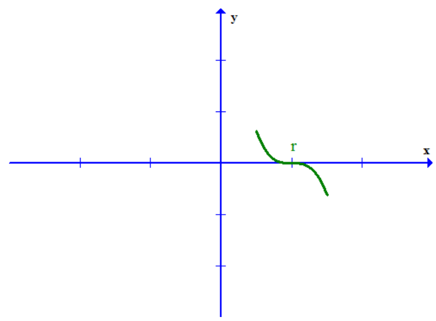


One of the theorems of algebra says that every polynomial can be factored in such a way that the only factors are :

(a) a number (the leading coefficient)

(b) $(x - r)^n$, where r is a zero with multiplicity n

(c) $(x^2 + bx + c)^m$, where $x^2 + bx + c$ is prime

Remark: This theorem says that such factorization is possible, but it does not say how to obtain such factorization.

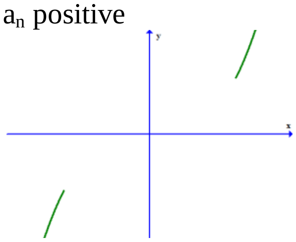
End behavior of a polynomial function

When |x| is large (x is large positive or large negative), then the graph of

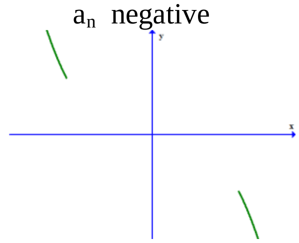
$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ behaves like the graph of $y = a_n x^n$, where a_n is the leading coefficient and n is the degree of f(x).

n- odd

a_n positive

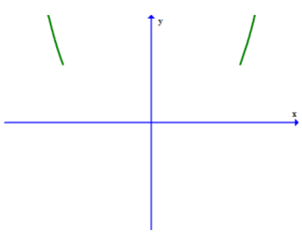


a_n negative

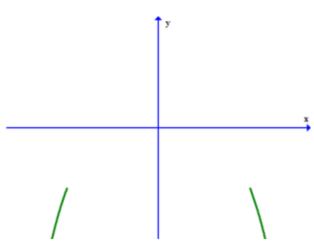


n- even

a_n positive



a_n negative



Example: Determine the degree and the leading coefficient of the polynomial function

$f(x) = 3(x-2)(x+4)^2(x^2+2)^3$. Give the equation of the power function that the function f behaves like for x with large absolute value.

$$f(x) = 3(x-2)(x+4)^2(x^2+2)^3$$

This polynomial is already factored.

The leading coefficient is 3.

The degree can be obtained by adding the highest exponents of x from each factor

factor	3	x-2	$(x+1)^2$	$(x^2+2)^3$
degree	0	1	2	6

Degree = 1 + 2+ 6 = 9

Therefore, for large |x|, f(x) behaves like $y = 3x^9$.

Sketching the graph of a polynomial function

(I) use transformations, when possible

(II) If the transformations cannot be performed, use the information above to sketch the graph

Example: Graph $f(x) = 2(x-1)^5 + 3$.

We can graph this function using transformations

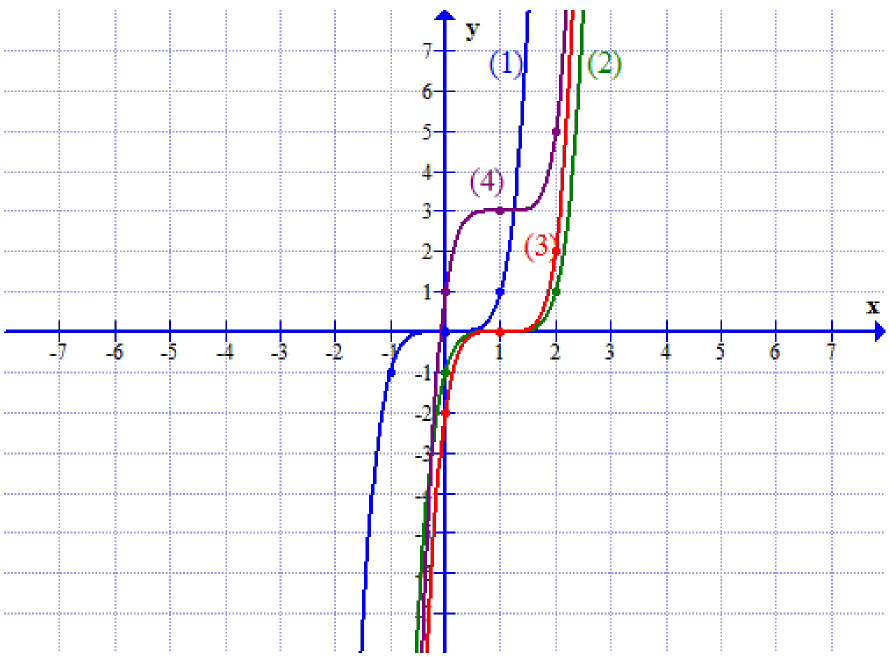
The order of transformations is as follows

1) graph basic function $y = x^5$

2) Shift the graph 1 unit to the right to obtain $y = (x-1)^5$

3) Stretch the graph by a factor of 2, to obtain $y = 2(x-1)^5$

4) Shift the graph up by 3 units to obtain $y = 2(x-1)^5 + 3$



Example: Graph $f(x) = 2(x-3)(x+4)^3$

Transformations cannot be used

(i) Determine the zeros, if any, of f, their multiplicity and the behavior of graph near each zero

$$f(x) = 2(x-1)^2(x+2)^3$$

zeros	1	- 2
multiplicity	2	3
behavior of graph	touches x-axis at 1	Crosses x-axis at -2 like a cubic function

(ii) Determine the end behavior of f

We need the leading coefficient and the degree of f(x)

$$f(x) = 2(x-3)(x+4)^3$$

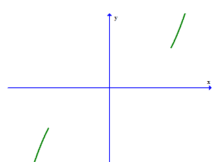
Leading coefficient = 2

Degree:

factor	2	$(x-3)^2$	$(x+4)^3$
degree	0	2	3

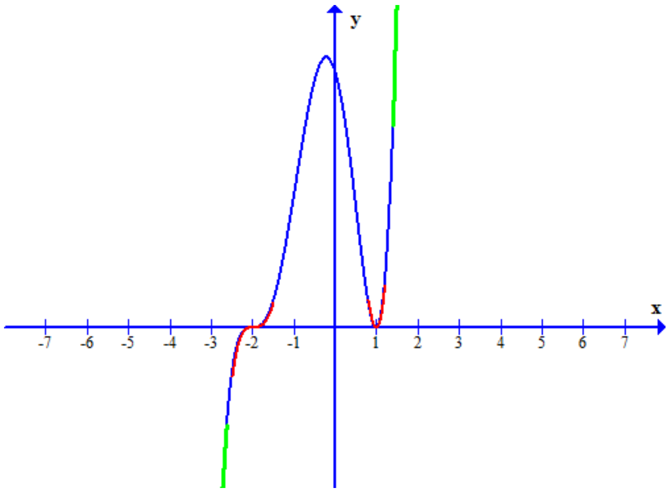
Degree of f = 0+2+3 = 5

f(x) behaves as $y = 2x^5$ for large |x|



(iii) Use the information from (i) and (ii) to draw the graph.

The graph will start in the third quadrant (green piece). It will continue to the first zero (-2) at which it will cross the x-axis like a cubic function (red piece). The graph will increase for a while, but then it will have to turn to reach the second zero(1), at which it will touch the x-axis (red piece). Since there are no more zeros, the graph will continue, eventually to reach green piece that depicts its behavior for the large positive x.



Example: Graph $f(x) = x^2(x^2+1)(x+4)^2$

Note that $x^2 + 1$ is always positive, so f(x) can be zero only when x = 0 or x = -4

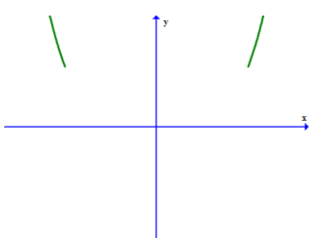
zeros	0	-4
multiplicity	2	2
behavior of graph	touches x-axis at 1	Touches x –axis at - 4

End behavior:

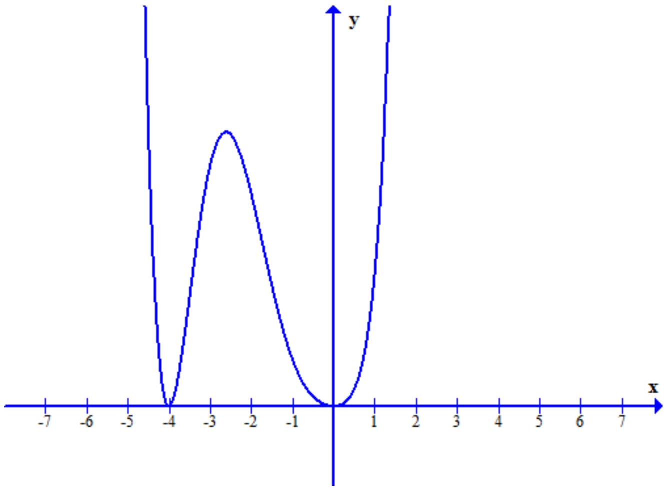
Leading coefficient = 1

factor	x^2	$x^2 + 1$	$(x+4)^2$
degree	2	2	2

Degree of f(x) = 2 + 2+ 2 = 6 and f behaves like $y = x^6$ for large |x|, that is, it looks as below



The graph of the function is below



Remarks:

- We don't have enough information to know at what points exactly the graph will change the direction (these points are called **turning points**). (You can learn this in Calculus.) But, we know that **there are at most (n-1) turning points** (n is the degree of the polynomial)

- Though we know how the graph of a polynomial function behaves for large positive and negative values and close to its zeros, we need Calculus to determine its behavior in- between. In this course however, we'll assume that nothing extraordinary takes place there.

5.2 Properties of Rational functions

A **rational function** is a function of the form

$$f(x) = \frac{polynomial}{polynomial} = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_k x^k + b_{k-1}x^{k-1} + \cdots + b_1 x + b_0}$$

Example

$$f(x) = \frac{3x^4 - x^2 - 2x + 5}{-x^2 + 4x + 1}$$

The **domain** of a rational function is the set of all real numbers except those x, for which q(x) = 0

To **find the domain**: (i) solve q(x) = 0

(ii) Write Df = {x | q(x) ≠ 0}

Example: Find the domain of $f(x) = \dfrac{3x^4 - x^2 - 2x + 5}{-x^2 + 4x + 1}$

(i) Solve: denominator = 0

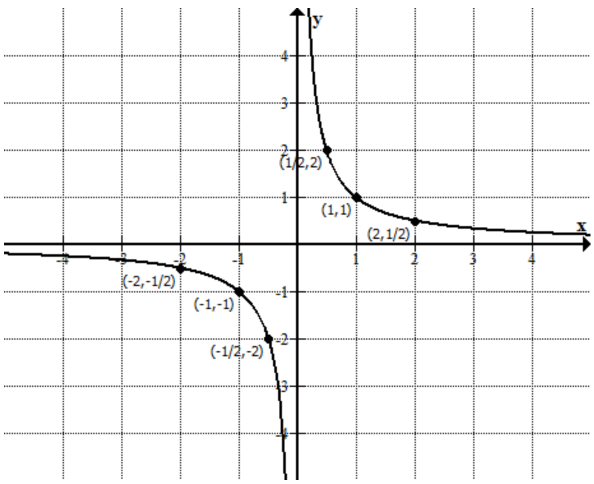
$$-x^2 + 4x + 1 = 0$$

$$x = \frac{-4 \pm \sqrt{4^2 - 4(-1)(1)}}{2(-1)} = \frac{-4 \pm \sqrt{20}}{-2} = \frac{-4 \pm 2\sqrt{5}}{-2} = \frac{-2(2 \mp \sqrt{5})}{-2} = 2 \pm \sqrt{5}$$

(ii) Df = {x|x $\neq 2 \pm \sqrt{5}$ } = $(-\infty, 2-\sqrt{5}) \cup (2-\sqrt{5}, 2+\sqrt{5}) \cup (2+\sqrt{5}, +\infty)$

Recall the graphs of $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{1}{x^2}$

$f(x) = \dfrac{1}{x}$



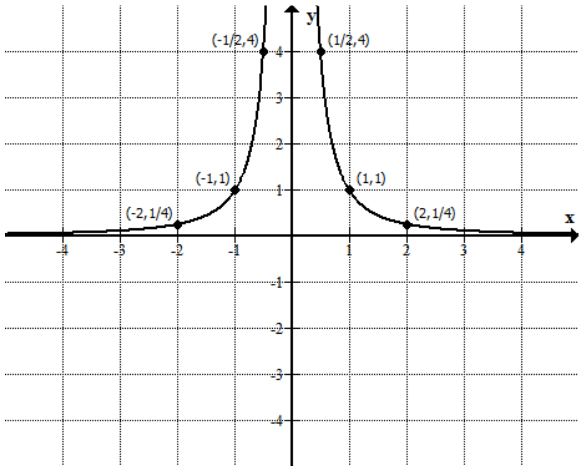
Note that when x is small (close to 0 on the x-axis) then its reciprocal is a large number (positive when x is positive and negative when x is negative). For example if x = 0.0001 then 1/x = 10, 000;

if x = - 0.000001 then 1/x = - 1,000,000. The smaller x, the larger 1/x. In mathematics we indicate that fact by saying that when x > 0 approaches 0, then 1/x = f(x) increases without bound and write

as x $\rightarrow 0^+$, then f(x) $\rightarrow +\infty$ or $\lim_{x \to 0^+} f(x) = +\infty$. Note that in such a case, the point (x, f(x)) on the graph approaches the y-axis. We say that the y-axis is a **vertical asymptote**.

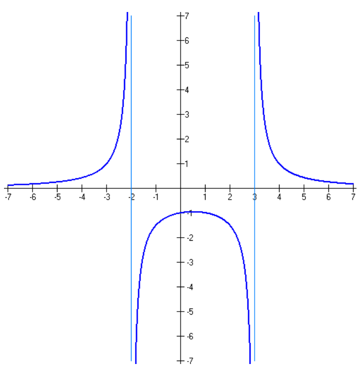
If on the other hand x $\rightarrow +\infty$ (that is, x increases without bound) than the values 1/x become smaller and smaller (if x = 10,000, then 1/x = 0.0001; if x = 1,000,000 then 1/x = 0.000001) so we say that as x $\rightarrow +\infty$, then 1/x = f(x) $\rightarrow$0 or that $\lim_{x \to +\infty} f(x) = 0$. Note that in this case, the point (x, f(x)) on the graph of the function approaches the x-axis. We say that the x-axis is a **horizontal asymptote**.

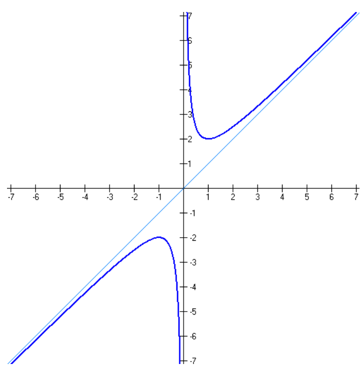
$$g(x) = \frac{1}{x^2}$$

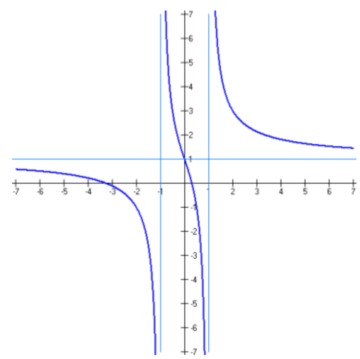


A rational function often has asymptotes: vertical and/or horizontal/oblique.

Informally speaking, an **asymptote** is a straight line (vertical, horizontal or slanted) toward which the graph comes near.



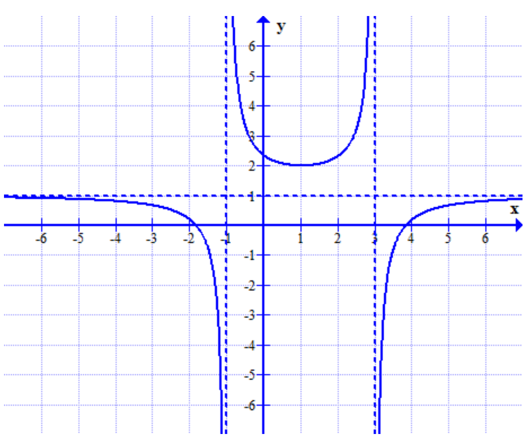




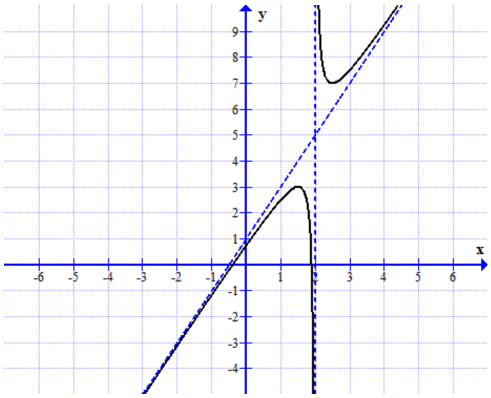
Vertical and horizontal asymptotes *vertical and oblique asymptotes* *vertical and horizontal asymptotes*

Example: Given the graph of a function, find the domain and give the equations of any asymptotes

a)



Domain $=\{x|x \neq -2,3\}$; vertical asymptotes: x =-2, x = 3; horizontal asymptote: y = 1 b)



Domain = {x| x ≠ 2} ; vertical asymptote: x = 2; slanted asymptote : y = 2x + 1 (line passes through (0,1) and (2,5))

How to find asymptotes

Vertical: 1. Reduce f(x) to the lowest terms:

(i) factor completely the numerator and the denominator;

(ii) cancel common factors

2. Solve the equation: denominator = 0

3. If x = r is a solution found in 2, then the line x = r is a vertical asymptote

Horizontal:

a) if *the degree of the numerator* < *the degree of the denominator*, then the line **y = 0** is the horizontal asymptote

(b) if *the degree of the numerator* = *the degree of the denominator*, then the line **y =** $\frac{a_n}{b_k}$ is the horizontal asymptote

(c) if *the degree of the numerator* > *the degree of the denominator*, then the graph does **no**t have a **horizontal asymptote**, however

Oblique: (d) if *the degree of the numerator = 1 + the degree of the denominator*, then the line

y = (quotient obtained by dividing the numerator by the denominator)

is an oblique(slanted) asymptote.

Remarks: 1. A rational function can have only **one horizontal/oblique** asymptote, but **many vertical** asymptotes.

2. If a rational function has a horizontal asymptote, then it **does not** have an oblique one.

3. The graph of a rational function can cross a horizontal/oblique asymptote, but does not cross a vertical asymptote

4. Horizontal/oblique asymptotes describe the behavior of function for x with large absolute value (the end behavior); vertical asymptotes describe the behavior of function near a point.

Example: Find the asymptotes for the following functions

a) $f(x) = \dfrac{3x+5}{2x-6}$

Vertical asymptote: 1) f is in lowest terms

2) 2x-6 = 0
2x = 6
x = 3
3) vertical asymptote: x = 3

Horizontal/oblique asymptote:

degree of numerator (1) = degree of the denominator(1), $y = \dfrac{3}{2}$ is the horizontal asymptote

b) $f(x) = \dfrac{2x^2+5x-1}{3x^3-6x^2}$

Vertical asymptote: 1) $f(x) = \dfrac{2x^2+5x-1}{3x^3-6x^2} = \dfrac{2x^2+5x-1}{3x^2(x-2)}$ is in lowest terms (numerator can't be factored)

2) $3x^3 - 6x^2 = 0$
$3x^2(x-2) = 0$
$x^2 = 0$ or $x - 2 = 0$
x = 0 or x = 2
3) vertical asymptotes : x = 0, x = 2

Horizontal/oblique asymptote:

degree of numerator (2) < degree of the denominator(3), y = 0 is the horizontal asymptote

c) $f(x) = \dfrac{3x^5-1}{x^2+2}$

Vertical asymptote: 1) f(x) is in lowest terms (the denominator cannot be factored)

2) $x^2 + 2 = 0$
$x^2 = -2$ (not possible)
no solution
3) vertical asymptotes : none

Horizontal/oblique asymptote:

degree of numerator (5) > degree of the denominator(2), there is *no horizontal* asymptote

degree of numerator (5) $\neq$ 1 + degree of the denominator(2), there is *no oblique* asymptote

d) $f(x) = \dfrac{3x^3-4x^2+1}{x^2-2}$

Vertical asymptote: 1) f(x) is in lowest terms

2) $x^2 - 2 = 0$
$x^2 = 2$
$x = \pm\sqrt{2}$

3) vertical asymptotes : x = - $\sqrt{2}$, x = $\sqrt{2}$

Horizontal/oblique asymptote:
degree of numerator (3) > degree of the denominator(2), there is *no horizontal* asymptote

degree of numerator (3) = 1 + degree of the denominator(2), there is an oblique asymptote

$$\begin{array}{r l} & \underline{3x - 4} \\ x^2 - 2 \,\big)\, & 3x^3 - 4x^2 + 1 \\ & \underline{-3x^3 \qquad + 6x} \\ & \quad -4x^2 + 6x + 1 \\ & \quad \underline{4x^2 \qquad - 8} \\ & \qquad\quad 6x - 7 \end{array}$$

Oblique asymptote: y = 3x -4

Remark: When looking for a vertical asymptote, it is **important** to make sure that the function is reduced to the lowest terms. **To see why,** consider function $f(x) = \dfrac{x^2 - 4}{x+2}$. Note that

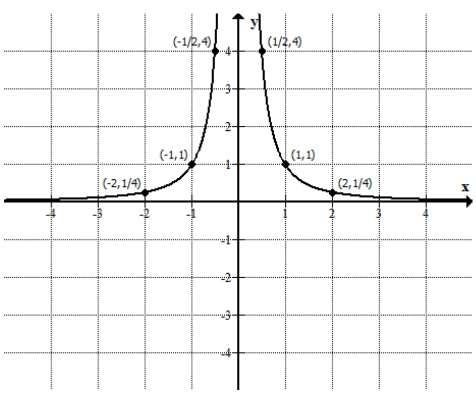
$f(x) = \dfrac{x^2 - 4}{x+2} = \dfrac{(x-2)(x+2)}{(x+2)} = (x-2), \quad if\ x \neq -2.$ The graph of *f*(x) is therefore a straight line y = x-2, with a hole at (-2, -4), hence has no asymptote.

Graphing rational functions

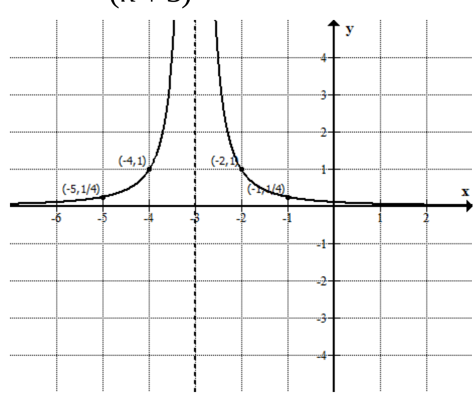
We can graph some rational functions using transformations

Example: Use transformations to graph $f(x) = \dfrac{2}{(x+3)^2} - 1$

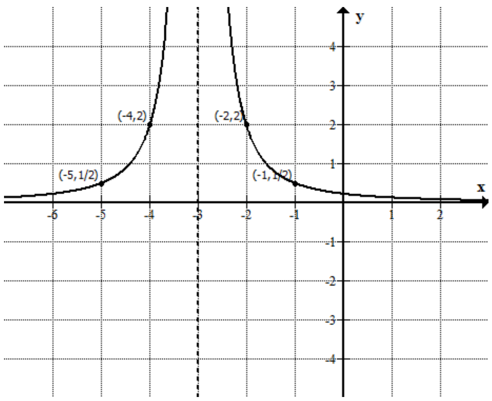
Basic function: $y = \dfrac{1}{x^2}$

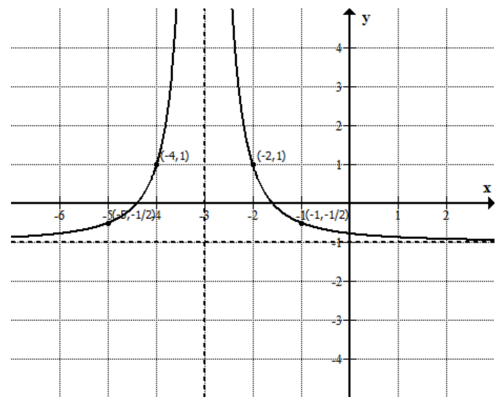


$y = \dfrac{1}{(x+3)^2}$ (shift left by 3)



$y = \dfrac{2}{(x+3)^2} = 2 \cdot \dfrac{1}{(x+3)^2}$ (vertical stretch) $\qquad$ $y = \dfrac{2}{(x+3)^2} - 1$ (Shift down by 1)





5.3 Sketching the graph of a rational function $f(x) = \dfrac{p(x)}{q(x)}$

1. Find the **domain:** (i) solve q(x) = 0
   (ii) Df = {x | q(x) $\neq$ 0}

2. Find x- and y-**intercepts:** y-intercept: y = f(0)
   x- intercepts: numerator = 0

3. Find **vertical asymptotes**, if any

Remark: If x = r is excluded from the domain and x = r is not a vertical asymptote, then the graph of *f* will pass through the point (r, "reduced"f (r)) but the point itself will not be included. We put an open circle around that point
The graph of *f* has a "hole" at x = r

4. Find the **horizontal/oblique asymptote**, if any.

5. Find the points where the graph **crosses** the horizontal/oblique **asymptote** y = mx +b
   (i) solve the equation f(x) = mx + b

6. Check for **symmetries**
   (i) If f(-x) = f(x), then the graph is symmetric about y- axis;
   (ii) If f(-x) = - f(x), then the graph is symmetric about the origin

Remark: If the graph is symmetric then only graph function for x >0 and use symmetry to graph the corresponding part for x<0

7. Make the **sign chart** for the "reduced" f(x)
   (i) plot x-intercepts and points excluded from the domain on the number line; these points divide the number line into a finite number of test intervals
   (ii) choose a point in each test interval and compute the value of f at the test point
   (iii) based on the sign of f at the test point, assign the sign to each test interval

Remark: When f(x) > 0, then the graph of f is above the x-axis.
When f(x) < 0, then the graph is below the x-axis

8. Sketch the **graph** of f using 1)-7):
   (i) Draw coordinate system and draw all asymptotes using a dashed line
   (ii) plot the intercepts, points where the graph crosses the horizontal/oblique asymptote and the points from the table in step 7.
   (iii) join the points with a continuous curve taking into consideration position of the graph relative to the x-axis (step 7) and behavior near asymptotes.

Example: Graph $f(x)=\dfrac{x^2+x-12}{x^2-4}$

1) Domain: $x^2 - 4 = 0$

$x^2 = 4$

$x = 2,\ x = -2$

$Df = \{x|x \neq -2, 2\}$

2) y –intercept: y = f(0) = (-12)/(-4)= 3

x- intercepts: $x^2 + x - 12 = 0$

$(x+4)(x-3) = 0$

$x = -4$ or $x = 3$

3) Vertical asymptotes $f(x)=\dfrac{x^2+x-12}{x^2-4}=\dfrac{(x+4)(x-3)}{(x-2)(x+2)}$

$(x-2)(x+2) = 0$

$x = 2 \quad x = -2$

vertical asymptotes: x = -2, x = 2

4) Horizontal/oblique asymptotes

Degree of numerator(2) = degree of the denominator(2), y = 1/1= 1 is the horizontal asymptote

5) Intersection with asymptote: f(x) = 1

$$\frac{x^2+x-12}{x^2-4}=1$$

$$x^2+x-12=x^2-4$$

$$x=8$$

The graph crosses the horizontal asymptote at x = 8, that is at the point (8,1)

6) Symmetries:

$$f(x)=\frac{x^2+x-12}{x^2-4}$$

$$f(-x)=\frac{(-x)^2+(-x)-12}{(-x)^2-4}=\frac{x^2-x-12}{x^2-4}$$

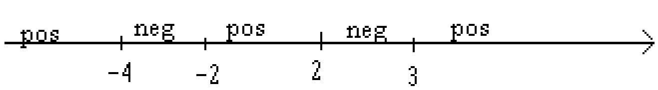
f(x) is not the same as f(-x), so f is not even and therefore not symmetric about y-axis

$$f(-x)=\frac{(-x)^2+(-x)-12}{(-x)^2-4}=\frac{x^2-x-12}{x^2-4}$$

$$-f(x)=-\frac{x^2+x-12}{x^2-4}=\frac{-x^2-x+12}{x^2-4}$$

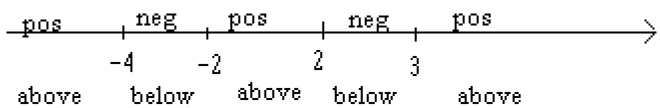
f(-x) and –f(x) are not the same so, f is not odd and therefore not symmetric about the origin

7)

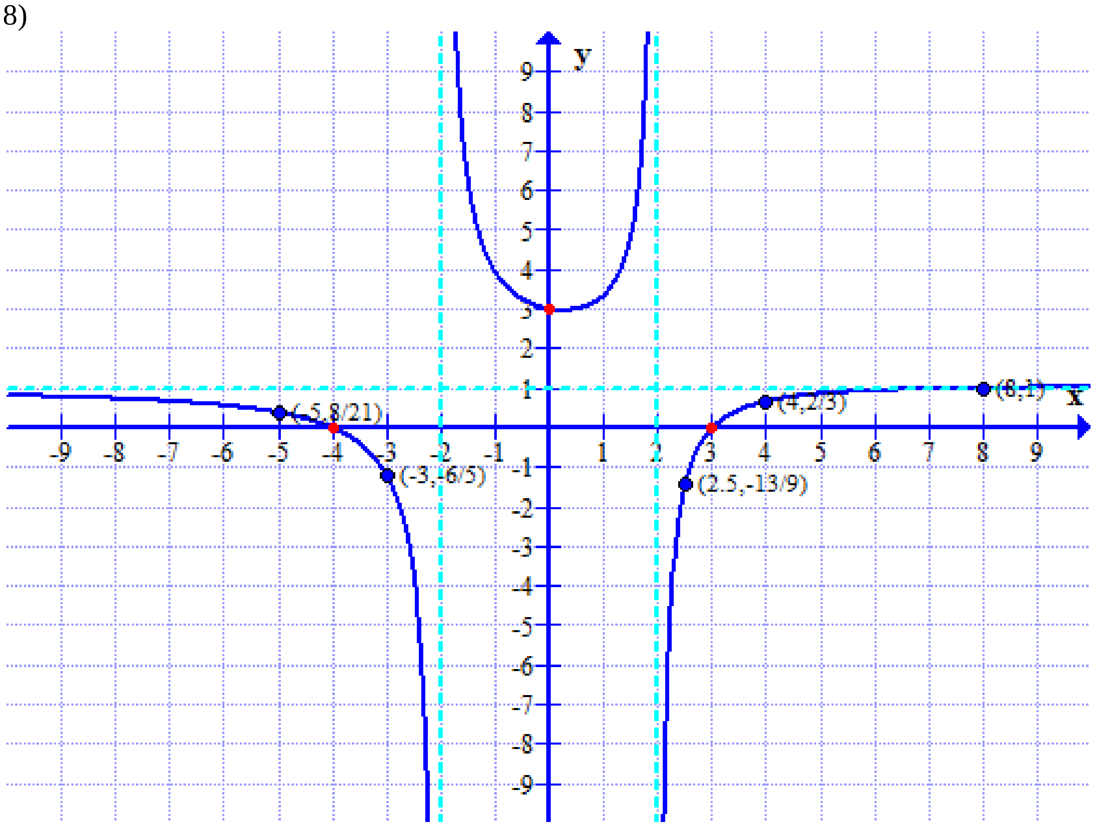


x	$f(x)=\dfrac{x^2+x-12}{x^2-4}$
-5	$\dfrac{(-5)^2+(-5)-12}{(-5)^2-4}=\dfrac{8}{21}$ *positive*
-3	$\dfrac{(-3)^2+(-3)-12}{(-3)^2-4}=-\dfrac{6}{5}$ *negative*
0	3 *positive*

2.5	$\frac{2.5^2+2.5-12}{2.5^2-4}=\frac{13}{9}$ *negative*
4	$\frac{4^2+4-12}{4^2-4}=\frac{2}{3}$ *positive*



8)



5.4 Solving polynomial and rational inequalities

Solving by graphing

To **solve** an inequality f(x) > 0 (< 0, $\geq$ 0, $\leq$ 0) **by graphing**:

(i) Graph function y = f(x). Make sure to compute exactly the x-intercepts (solve f(x) = 0)

(ii) Read the solutions from the graph.

To solve f(x) > 0, find the intervals (the x values) on which the graph is above x-axis but not on x axis.

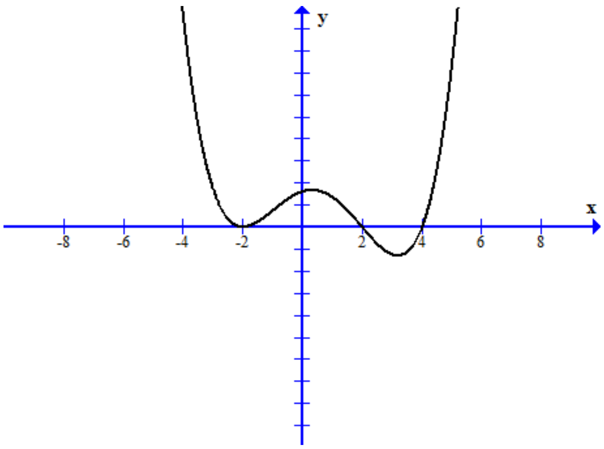
To solve f(x) < 0, find the intervals (the x values) on which the graph is below x-axis, but not on the x-axis.

To solve f(x) $\geq$ 0, find intervals (the x values) on which the graph is above or on the x-axis.

To solve f(x) $\leq$ 0, find intervals (the x values) on which the graph is below or on the x-axis.

Example: The graph of a function f is given below. Use the graph to solve given inequality:

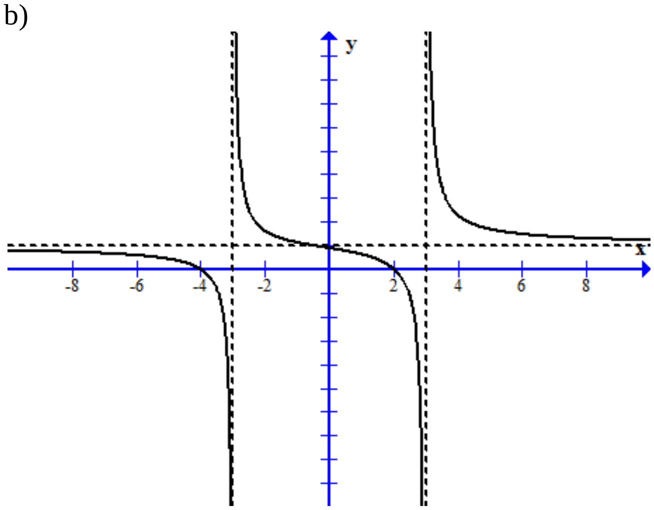
a)



f(x) > 0 for x in (-∞,-2) ∪ (-2,2) ∪ (4, +∞)

f(x) $\geq$ 0 for x in (-∞,2] ∪ [4, +∞)

b)

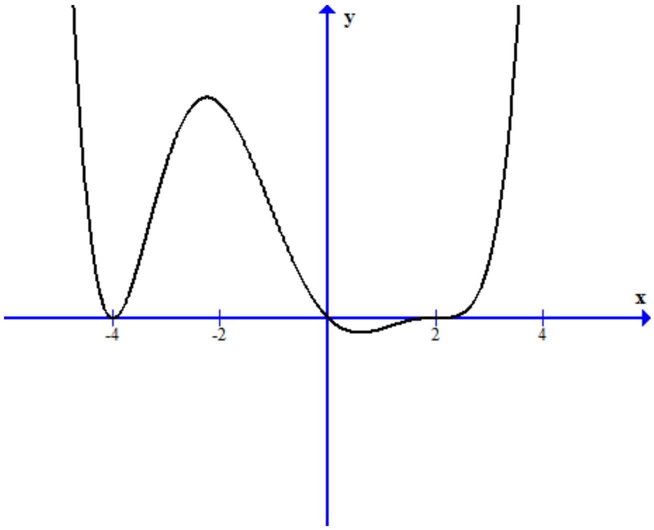


f(x) < 0 for x in (-4,-3)∪(2,3)

f(x) $\leq$ 0 for x in [-4,-3) ∪ [2, 3)

Example: Solve $2x(x-2)^3(x+4)^2 > 0$

Let $f(x) = 2x(x-2)^3(x+4)^2$.
f is a polynomial with zeros: 0, 2, -4. The multiplicities of zeros are 1, 3, 2 respectively. The graph will cross the x axis at 0 and 2 and touch the x-axis at -4. For large |x|, f(x) behaves line $y = 2x^6$. The graph of f(x) is below:



Now, f(x) > 0 when x belongs to (-∞,-4) ∪ (-4,0) ∪ (2, +∞).
Therefore the solution of $2x(x-2)^3(x+4)^2 > 0$ is (-∞,-4) ∪ (-4,0) ∪ (2, +∞).

Algebraic methods
Solving a Polynomial Inequality: polynomial > 0 ($\geq$ 0, < 0, $\leq$ 0)

(i) Write the inequality in the standard form (0 on the right hand side)
(ii) Find the zeros of the polynomial that is solve the equation : polynomial = 0
(iii) Plot the zeros on the number line
(iv) The zeros divide the number line into a finite number of intervals on which the polynomial has the same sign. Choose a number in each interval (a test point) and evaluate the value of the polynomial at each number.
(v) If the value of the polynomial at the chosen number is positive (> 0), then the polynomial is positive on the whole interval
If the value of the polynomial at the chosen number is negative (< 0), then the polynomial is negative on the whole interval
(vi) Choose, as the solution, the intervals on which the polynomial has a desired sign. Use interval notation. Include the endpoints only when the original inequality is $\leq$ or $\geq$.
If there are two separate intervals on which the polynomial has a desired sign, use the union sign, ∪, between the intervals.

Example: Solve $x^3 \geq x$

(i) $x^3 \geq x$
$x^3 - x \geq 0$

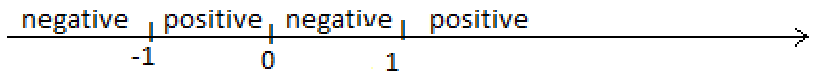
(ii) $x^3 - x = 0$
$x(x^2 - 1) = 0$
$x = 0$ or $x^2 - 1 = 0$
$x^2 = 1$
$x = \pm\sqrt{1} = \pm 1$

(iii)

(iv)

interval	Test point	Value of x^3-x at the test point
(-∞, -1)	-2	$(-2)^3 - (-2) = -8 + 2 = -6$ negative
(-1,0)	- 0.5	$(-0.5)^2 - (-0.5) = -.125 + .5 = .375$ positive
(0, 1)	0.5	$(.5)^3 - (.5) = -.375$ negative
(1, ∞)	2	$2^3 - 2 = 6$ positive

(v)



(vi) Since the inequality is $x^3 - x \geq 0$, we choose the intervals on which the polynomial is positive and include the endpoints. There are two intervals, so we use ∪.

Solution: $[-1,0] \cup [1,\infty)$

Solving a Rational Inequality: $\frac{P}{Q} > 0 \quad (\geq 0, < 0, \leq 0)$, P, Q are polynomials

(i) Write the inequality in the standard form $\frac{P}{Q} > 0 \quad (\geq 0, < 0, \leq 0)$ (0 on the right hand side)

(ii) Solve the equations: P = 0 and Q = 0

(iii) Plot the solutions on the number line. Place open circle at each solution of Q = 0; those numbers cannot ever be included in the solution set (they make the denominator zero and are out of the domain)

(iv) The solutions divide the number line into a finite number of intervals. A rational function will have a constant sign in each interval. Choose a number in each interval and evaluate the value of the expression $\frac{P}{Q}$ at each number.

(v) If the value of $\frac{P}{Q}$ is positive (> 0), then $\frac{P}{Q}$ is positive on the whole interval

If the value of $\frac{P}{Q}$ is negative (< 0), then $\frac{P}{Q}$ is negative on the whole interval

(vi) Choose, as the solution, the intervals on which $\frac{P}{Q}$ has a desired sign. Use interval notation. Include the endpoints only when the original inequality is $\leq$ or $\geq$. Remember to **never include** the endpoint with an open circle!

If there are two or more such intervals, use the union sign, ∪.

Example: Solve $\frac{x+2}{x-4} \geq 2$

(i)

$$\frac{x+2}{x-4} \geq 2$$

$$\frac{x+2}{x-4} - 2 \geq 0$$

$$\frac{x+2}{x-4} - \frac{2(x-4)}{x-4} \geq 0$$

$$\frac{x+2-2x+8}{x-4} \geq 0$$

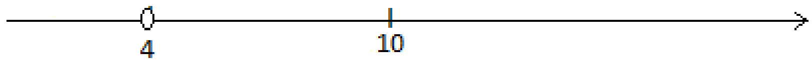
$$\frac{-x+10}{x-4} \geq 0$$

(ii) Numerator = 0 denominator = 0

$-x+10 = 0$ $x - 4 = 0$

$x = 10$ $x = 4$

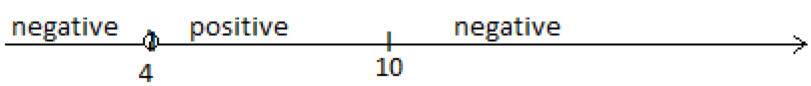
(iii)



(iv)

interval	Test point	Value of $\frac{-x+10}{x-4}$ at the test point
(-∞ 4)	0	$\frac{-0+10}{0-4} = \frac{10}{-4}$ negative
(4, 10)	5	$\frac{-5+10}{5-4} = \frac{5}{1}$ positive
(10, +∞)	11	$\frac{-11+10}{11-4} = \frac{-1}{7}$ negative

(v)



(vi) Since the inequality is $\frac{-x+10}{x-4} \geq 0$, we choose the intervals on which $\frac{-x+10}{x-4}$ is positive and include endpoints that do not have an open circle .

Solution: (4, 10]