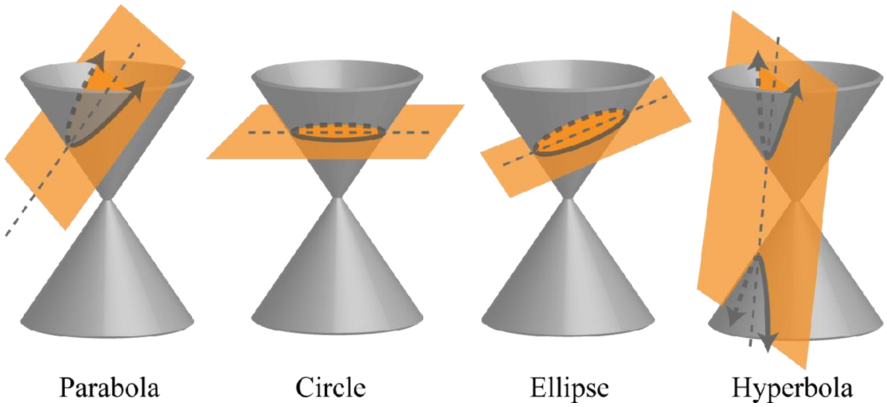
10. Conic sections (conics)

Conic sections are formed by the intersection of a plane with a right circular cone. The type of the curve depends on the angle at which the plane intersects the surface



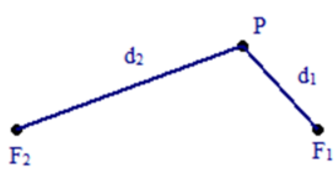
A circle was studied in algebra in sec 2.4. We will discuss the remaining 3 conics.

10.1 Ellipse

Definition:
An ellipse is the set of points P on the plane whose sum of the distances from two different fixed points F_1 and F_2 is constant, that is,

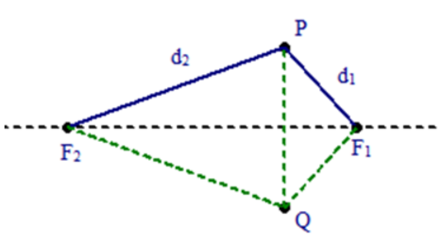
$$\text{dist}(P, F_1) + \text{dist}(P, F_2) = 2a$$

where a is a positive constant.



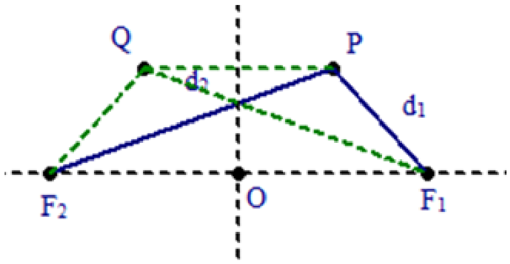
$d_1 + d_2 = 2a$

Note that the point Q, symmetric to P with respect to the line through F_1 and F_2, also satisfies this equation, and hence, is on the ellipse.

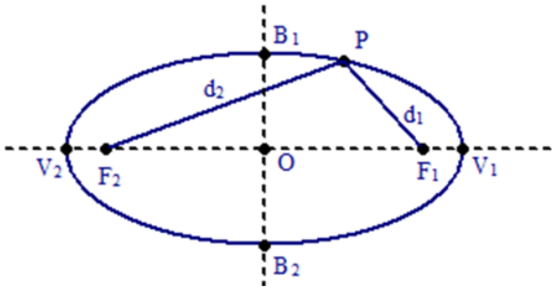


Moreover the point Q symmetric to P with respect to the line perpendicular to F_1F_2 and passing through the midpoint of the segment F_1F_2 is also on the ellipse since

dist (Q,F_1)+dist(Q, F_2) = dist(P, F_2) + dist(P, F_1)



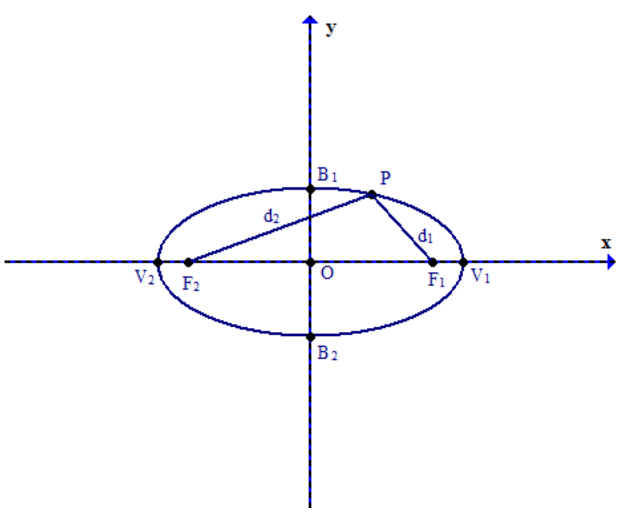
Therefore, an ellipse is symmetric with respect to the line F_1F_2 as well as the line perpendicular to F_1F_2 and passing through point O. Point O, which is the midpoint of the segment F_1F_2, is called the **center** of the ellipse.
Here is the graph of this ellipse:



The points F_1 and F_2 are called **foci** of the ellipse. The points V_1 and V_2 are called **vertices** of the ellipse. The segment V_1V_2 is called the **major axis** and the segment B_1B_2 is called **the minor axis** of the ellipse.

Let's note the **basic properties of an ellipse**:
- vertices, foci and the center of an ellipse lie on the same line
- the center of an ellipse is halfway between foci
- the center of an ellipse is halfway between vertices
-an ellipse is symmetric with respect to the line containing major axis and symmetric with respect to the line containing the minor axis
-the major axis is longer than the minor axis
- the vertices, foci and the center are on the major axis

Equation of an ellipse with the center at the origin and the foci along x- axis



Suppose that $F_1 = (c,0)$, $c > 0$. Because of the symmetry, $F_2 = (-c,0)$ and the length of the segment $F_1 F_2$ is 2c. Since in any triangle the sum of two sides must always be greater than the third side, in the triangle F_1PF_2, we must have $d_1+d_2 >$ dist (F_1, F_2), or $2a > 2c$, or $a > c$.

Let P =(x,y) be any point on the ellipse. Then, using the distance formula, we get
$d_1 = dist(P, F_1) = \sqrt{(x-c)^2 + (y-0)^2}$ and $d_2 = dist(P, F_2) = \sqrt{(x-(-c))^2 + (y-0)^2}$.

Therefore, since for ellipse, dist(P, F_1) + dist(P, F_2) = 2a, we have

$$\sqrt{(x-c)^2 + (y-0)^2} + \sqrt{(x-(-c))^2 + (y-0)^2} = 2a$$

or $$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a$$

This is the equation of an ellipse. Can it be simplified?

Let's eliminate the radicals.

Re-write: $\sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}$

Square both sides: $(x-c)^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2$

Expand the squares: $x^2 - 2xc + c^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2xc + c^2 + y^2$

Combine the like terms: $4a\sqrt{(x+c)^2 + y^2} = 4a^2 + 4xc$

Divide both sides by 4 and then square both sides again: $a^2\left((x+c)^2 + y^2\right) = a^4 + 2a^2xc + x^2c^2$

Remove the parentheses on the left $a^2x^2 + 2a^2xc + a^2c^2 + a^2y^2 = a^4 + 2a^2xc + x^2c^2$

Combine the like terms: $(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$

Since a >c, then $a^2 > c^2$ and therefore a^2-c^2 is a positive number. Let $b = \sqrt{a^2 - c^2}$ or $b^2 = a^2 - c^2$. Note that, b< a.

With such defined b, we can re-write the equation of an ellipse as $b^2x^2 + a^2y^2 = a^2b^2$. After dividing both sides of this equation by a^2b^2 and simplifying we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

This is the **standard equation of an ellipse** with center at the origin and the foci on the x-axis.

How are *a* and *b* related to the graph of an ellipse?

Notice that the x-intercepts of the graph are the vertices of the ellipse. Using the equation we can find the x-intercepts (make y =0 and solve for x)

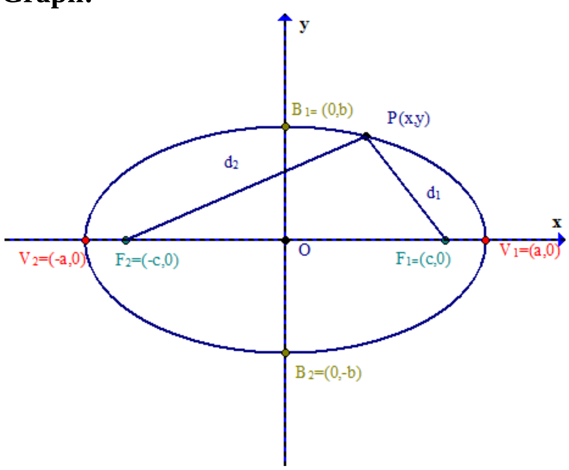
$$\frac{x^2}{a^2}+\frac{0^2}{b^2}=1$$
$$\frac{x^2}{a^2}=1$$
$$x^2=a^2$$
$$x=\pm\sqrt{a^2}=\pm a$$

Hence, the vertices are $V_1=(a,0)$, $V_2=(-a,0)$. Similarly, we can find the y-intercepts (set x = 0 and solve for y). The y-intercepts are $B_1=(0,b)$, and $B_2=(0,-b)$

Our findings are summarized below:

Standard equation of an ellipse with center at (0,0) and foci on the x-axis: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $b^2=a^2-c^2$

Center: (0,0)
Foci: (c,0), (-c,0);
Vertices: (a,0),(-a,0);
a= dist(vertex, center), b = dist (center, B); c = dist(focus, center) ; a >c, a > b
Graph:

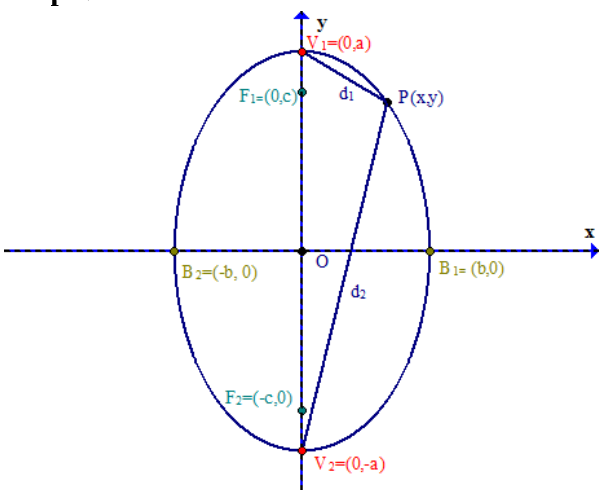


Equation of an ellipse with the center at the origin and the foci on the y- axis

Work similar to the above leads to the following facts:

Standard equation of an ellipse with center at (0,0) and foci on the y-axis: $\dfrac{x^2}{b^2}+\dfrac{y^2}{a^2}=1$, where $b^2=a^2-c^2$

Center: (0,0)
Foci: (0,c), (0,-c)
Vertices: (0,a),(0,-a)
a= dist(vertex, center), b = dist (center, B); c = dist(focus, center); a >c, a > b
Graph:



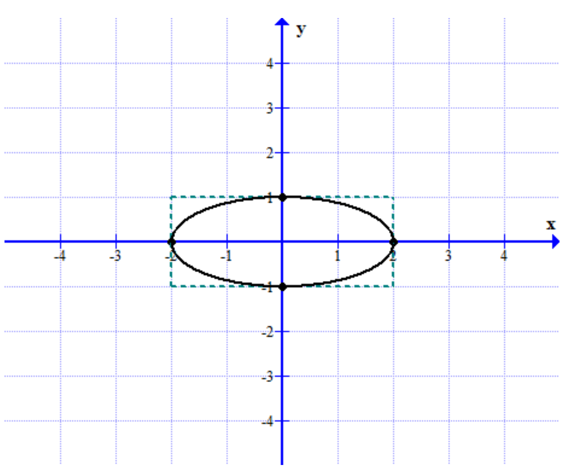
Example
Graph $x^2 + 4y^2 = 4$. Find the coordinates of the center, vertices and foci.

1. Write the equation in the standard form (divide each term by 4): $\dfrac{x^2}{4}+\dfrac{y^2}{1}=1$

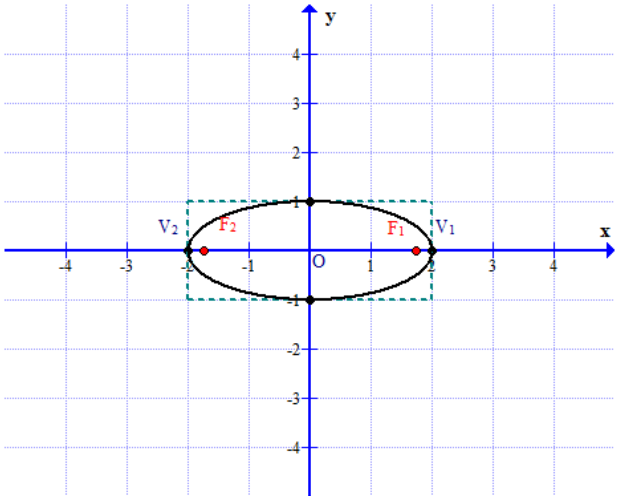
2. Find the x-intercepts: setting y = 0, we get $\dfrac{x^2}{4}=1$ *or* $x^2=4$ *or* $x=\pm\sqrt{4}=\pm2$

3. Find the y-intercepts: setting x = 0, we get $\dfrac{y^2}{1}=1$ *or* $y^2=1$ *or* $y=\pm\sqrt{1}=\pm1$

4. Plot the intercepts and draw a rectangle(using a dashed line) containing the intercepts with the sides parallel to the x- and y- axis
5. Draw an ellipse (an oval shape that fits within the rectangle)



The center is at the origin: (0,0)
Since the major axis is longer and the vertices are on the major axis, from the graph, we can read that the ***vertices are: (2,0), (-2,0).***
Since a = dist (center, vertex), then a = 2. Also, since the minor axis has the endpoint at (0,1), (0,-1), we conclude that b = dist(center, (0,1))= 1. Since $b^2 = a^2 - c^2$. *We get* $1 = 4 - c^2$ *or* $c^2 = 3$, *or* $c = \pm\sqrt{3}$. *Therefore the foci, which are on the major axis (hence on the x-axis)* ***are at*** $(\sqrt{3},0)$ ***and*** $(-\sqrt{3},0)$



Example
Find the equation of the ellipse with focus at (0,-3) and vertices at (0, ±6)

1. Since the vertices are (0,6) and (0,-6) , the center is halfway between the vertices (center is the midpoints of the segment with endpoints (0,6) and (0,-6)), hence, ***the center O is at (0,0)*** *(O =* $\left(\frac{0+0}{2}, \frac{6+(-6)}{2}\right)$*)*

2. Since the vertices and foci are on the y axis the equation of the ellipse is of the form: $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

3. since a = dist(center, vertex) = 6 and c = dist(center, focus) = 3, we have

$$b^2 = a^2 - c^2 = 6^2 - 3^2 = 36 - 9 = 27$$

Hence the equation of the ellipse with given properties is

$$\frac{x^2}{27} + \frac{y^2}{36} = 1$$

Equation of an ellipse with center at (h,k)
If an ellipse with center at (0,0) is shifted so its center moves to (h,k), its equation becomes

$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, if the foci and vertices are on the line parallel to the x-axis

And

$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$, if the foci and vertices are on the line parallel to the y-axis

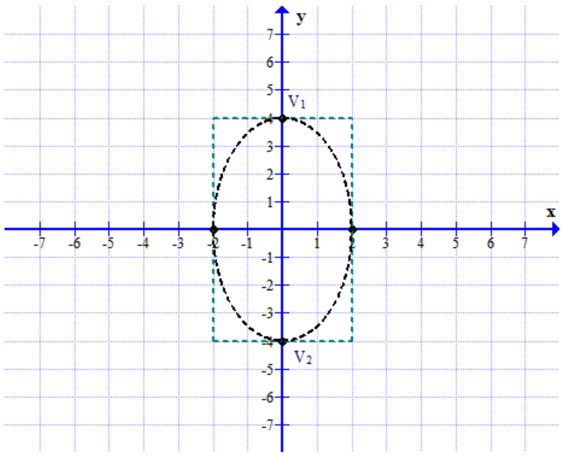
Example

Graph $\frac{(x-3)^2}{4}+\frac{(y+2)^2}{16}=1$. Find the coordinates of the center, foci and vertices.

Note that this is the ellipse $\frac{x^2}{4}+\frac{y^2}{16}=1$*, shifted, so its center is at (3,-2), hence shifted 3 units to the right and 2 units down.*

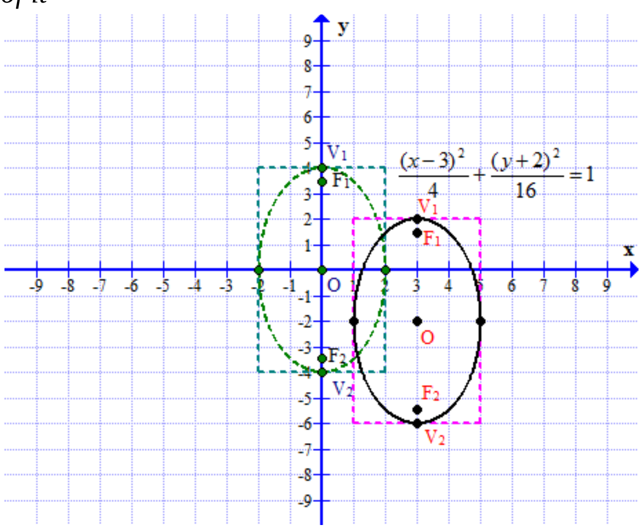
1. Graph, using a dashed line, an ellipse $\frac{x^2}{4}+\frac{y^2}{16}=1$

Center at (0,0); x-intercepts: $x=\pm\sqrt{4}=\pm 2$; *y-intercepts:* $y=\pm\sqrt{16}=\pm 4$



Note that a = dist(center, vertex) = 4; b = 2, c^2 = a^2 –b^2 = 16-4 = 12. So c = $\sqrt{12}=2\sqrt{3}$ *and the foci are* $(0,-2\sqrt{3})$ *and* $(0,2\sqrt{3})$

2. Shift the center, vertices and foci 3 units to the right and 2 units down. Draw the rectangle and the ellipse inside of it



3. Use the graph and the transformations to obtain the coordinates of the key points for the ellipse

$$\frac{(x-3)^2}{4}+\frac{(y+2)^2}{16}=1$$

Center: (3,-2)

Vertices: (3,2), 3,-6)

Foci: $(3,-2-2\sqrt{3})$ *and* $(3,-2+2\sqrt{3})$

Example

Write the given equation in the standard form. Determine the coordinates of the center, vertices and foci.

$$2x^2 + 3y^2 + 8x - 6y + 5 = 0$$

1. Re-write the equation so the x-terms and the y-terms are together; move the number to the right hand side

$$(2x^2 + 8x) + (3y^2 - 6y) = -5$$

2. From the x- group, factor out the coefficient of x^2; from the y-group- factor the coefficient of y^2

$$2(x^2 + 4x) + 3(y^2 - 2y) = -5$$

3. Complete the square in each group; add appropriate numbers to the right hand side

$$2(x^2 + 4x + 4) + 3(y^2 - 2y + 1) = -5 + 2\cdot 4 + 3\cdot 1$$

4. Write each group as a perfect square; add numbers on the right

$$2(x+2)^2 + 3(y-1)^2 = 6$$

5. Divide each term by the number on the right hand side (6) and simplify

$$\frac{2(x+2)^2}{6} + \frac{3(y-1)^2}{6} = \frac{6}{6}$$

$$\frac{(x+2)^2}{3} + \frac{(y-1)^2}{2} = 1$$

This is the standard equation of the ellipse. ***Center*** *is at (-2,1).*

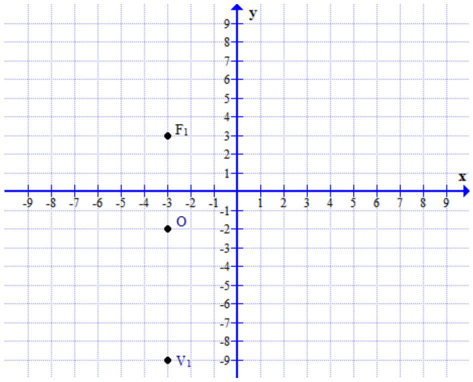
Hence, this ellipse is obtained by shifting the ellipse $\frac{x^2}{3} + \frac{y^2}{2} = 1$ *two units to the left and one unit up. The numbers in the denominators are 3 and 2 and since 3 > 2, the vertices of* $\frac{x^2}{3} + \frac{y^2}{2} = 1$ *are on the x-axis and* $a^2 = 3$ *and* $b^2 = 2$, *or* $a = \sqrt{3}$ *and* $b = \sqrt{2}$ *(remember that* $a^2 > b^2$*). Therefore, the* ***vertices*** *of this ellipse are at* $(\pm\sqrt{3}, 0)$. *Moreover, since* $c^2 = a^2 - b^2 = 3-2 = 1$, $c = 1$, *the foci are at* $(\pm 1, 0)$.

Therefore, for $\frac{(x+2)^2}{3} + \frac{(y-1)^2}{2} = 1$, *the vertices are* $(\pm\sqrt{3} - 2, 1)$ *and the foci are* $(\pm 1-2, 1)$ *or (-3,1) and (1,-1).*

Example

Find the equation of the ellipse with center at (-3,-2), focus at (-3,3) and vertex at (-3,-9). Graph the ellipse.

1. *Plot the given information in the coordinate system and use them to determine key values for the ellipse*

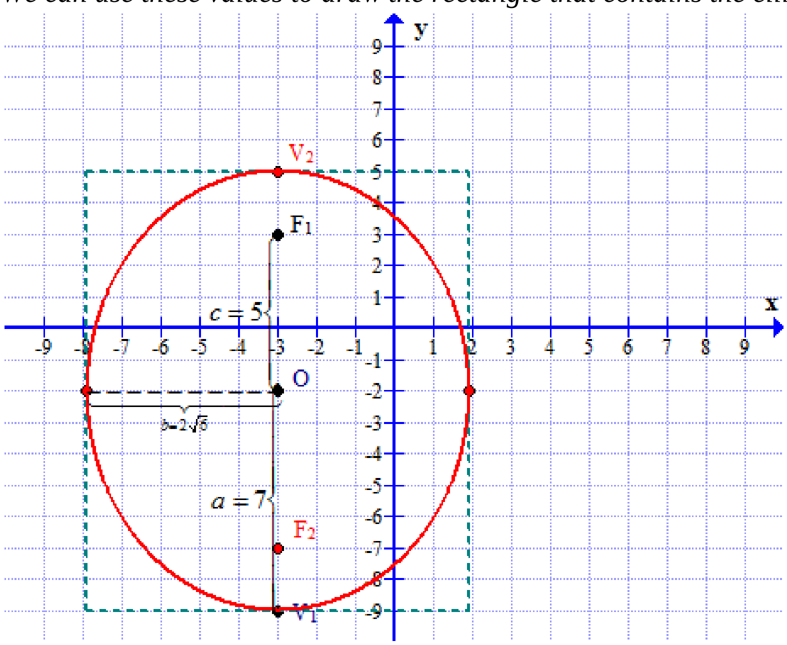


Since c = dist(center, focus), then c = 5 and the second focus is at (-3, -7). Since a = dist(center, vertex) , then a = 7 and the second vertex is at (-3,5). Also, $b^2 = a^2 - c^2 = 7^2 - 5^2 = 49-25=24$*, so* $b = \sqrt{24} = 2\sqrt{6}$ *.Vertices and foci are on the line parallel to the y-axis, therefore the standard equation of this ellipse will be of the form*

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1.$$

Using values that we determined, the equation is $\frac{(x-(-3))^2}{24} + \frac{(y-(-2))^2}{49} = 1$ *or* $\frac{(x+3)^2}{24} + \frac{(y+2)^2}{49} = 1$.

We can use these values to draw the rectangle that contains the ellipse and the ellipse itself.

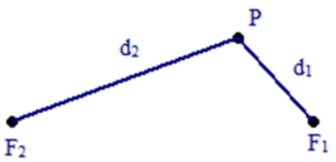


10.2 Hyperbola

Definition:
A **hyperbola** is the set of points P on the plane for which the difference of the distances from two different, fixed points F_1 and F_2 is constant, that is,

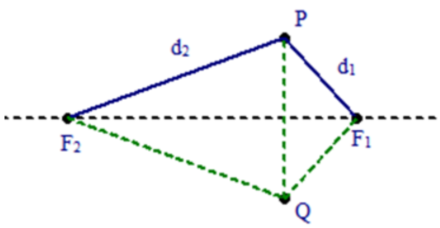
$$|\text{dist}(P, F_1) - \text{dist}(P, F_2)| = 2a$$

where a is a positive constant.

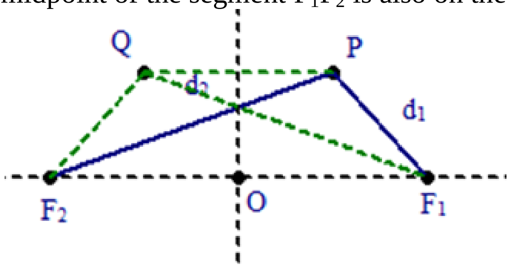


$d_2 – d_1 = 2a$

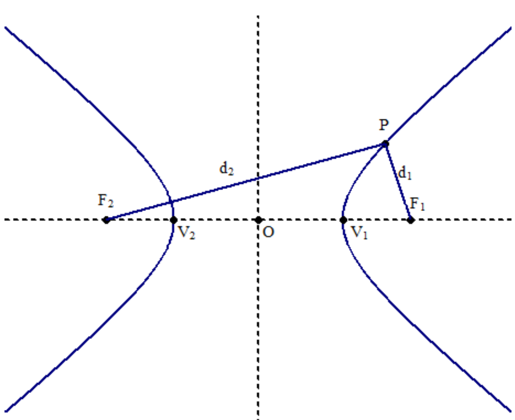
Note that the point Q, symmetric to P with respect to the line through F_1 and F_2, also satisfies this equation (since d_2= dist(Q, F_2) and d_1 = dist(Q, F_1) , and hence, is on the hyperbola.



Moreover the point Q symmetric to P with respect to the line perpendicular to F_1F_2 and passing through the midpoint of the segment F_1F_2 is also on the hyperbola since dist (Q,F_1) - dist(Q, F_2) = dist(P, F_2) - dist(P, F_1)



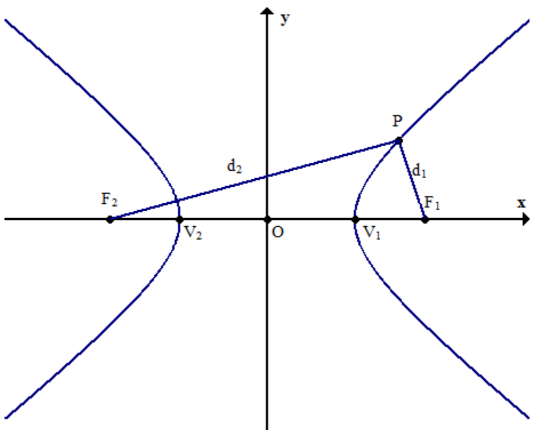
Therefore, a hyperbola is symmetric with respect to the line F_1F_2 as well as the line perpendicular to F_1F_2 and passing through point O. Point O, which is the midpoint of the segment F_1F_2, is called the **center** of the hyperbola. Here is the graph of this hyperbola:



The points F_1 and F_2 are called **foci** of the hyperbola. The points V_1 and V_2 are called **vertices** of the hyperbola. The line V_1V_2 is called the **transverse axis** and the line perpendicular to it and passing through the center O is called **the conjugate axis** of the hyperbola.

Let's note the **basic properties of a hyperbola**:
-hyperbola consists of two parts called branches
- vertices, foci and the center of a hyperbola lie on the same line(transverse axis)
- the center of a hyperbola is halfway between foci
- the center of a hyperbola is halfway between vertices
-a hyperbola is symmetric with respect to the transverse and conjugate axes

Equation of a hyperbola with the center at the origin and the foci on the x- axis



Suppose that $F_1 = (c,0)$, $c > 0$. Because of the symmetry, $F_2 =(-c,0)$ and the length of the segment $F_1 F_2$ is 2c. Since in any triangle the sum of two sides must always be greater than the third side, in the triangle F_1PF_2, we must have $d_1 + \text{dist}(F_1, F_2) > d_2$, or $\text{dist}(F_1,F_2) > d_2 - d_1$, which means that $2c > 2a$, or $c > a$.

Let P = (x,y) be any point on the hyperbola. Then, using the distance formula, we get
$d_1 = dist(P,F_1) = \sqrt{(x-c)^2 + (y-0)^2}$ and $d_2 = dist(P,F_2) = \sqrt{(x-(-c))^2 + (y-0)^2}$.

Therefore, since for hyperbola, $|\text{dist}(P, F_1) - \text{dist}(P, F_2)| = 2a$, we have

$$\left|\sqrt{(x-c)^2 + (y-0)^2} - \sqrt{(x-(-c))^2 + (y-0)^2}\right| = 2a$$

or $$\left|\sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2}\right| = 2a$$

This is the equation of a hyperbola. Can this equation be simplified?

Eliminate the absolute value first: $\sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} = \pm 2a$

Re-write: $\sqrt{(x-c)^2 + y^2} = \pm 2a + \sqrt{(x+c)^2 + y^2}$

Square both sides: $(x-c)^2 + y^2 = 4a^2 \pm 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2$

Expand the squares: $x^2 - 2xc + c^2 + y^2 = 4a^2 \pm 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2xc + c^2 + y^2$

Combine the like terms: $\pm 4a\sqrt{(x+c)^2+y^2} = 4a^2+4xc$

Divide both sides by 4 and then square both sides: $a^2\left((x+c)^2+y^2\right) = a^4+2a^2xc+x^2c^2$

Remove the parentheses on the left $a^2x^2+2a^2xc+a^2c^2+a^2y^2 = a^4+2a^2xc+x^2c^2$

Combine the like terms: $(a^2-c^2)x^2+a^2y^2 = a^2(a^2-c^2)$

Re-write: $-(c^2-a^2)x^2+a^2y^2 = -a^2(c^2-a^2)$

Since c >a, then $c^2 > a^2$ and therefore c^2-a^2 is a positive number. Let $b=\sqrt{c^2-a^2}$ or $b^2=c^2-a^2$.

With such defined b, we can re-write the equation of an hyperbola as $-b^2x^2+a^2y^2 = -a^2b^2$. After dividing both sides of this equation by $-a^2b^2$ and simplifying we get

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

This is the **standard equation of a hyperbola** with center at the origin and the foci on the x-axis.

How are *a* and *b* related to the graph of the hyperbola?

Notice that the x-intercepts of the graph are the vertices of the hyperbola. Using the equation we can find x-intercepts (make y =0 and solve for x)

$$\frac{x^2}{a^2}-\frac{0^2}{b^2}=1$$

$$\frac{x^2}{a^2}=1$$

$$x^2=a^2$$

$$x=\pm\sqrt{a^2}=\pm a$$

Hence, the vertices are V_1= (a,0), V_2=(-a, 0).

Note that if we try to find the y-intercepts (set x =0 and solve for y), we get the equation $y^2 = -b^2$, which has no real solutions. Hence this hyperbola has no y-intercepts.

However, the equation allows us to discover another property of a hyperbola, namely the existence of asymptotes. Note that we can re-write the equation (solve for y^2) as follows

$$\frac{y^2}{b^2}=\frac{x^2}{a^2}-1$$

$$y^2=b^2\left(\frac{x^2}{a^2}-1\right)=b^2\frac{x^2}{a^2}\left(1-\frac{1}{\frac{x^2}{a^2}}\right)=\frac{b^2}{a^2}x^2\left(1-\frac{a^2}{x^2}\right)$$

Solving for y gives $y=\pm\frac{b}{a}x\sqrt{\left(1-\frac{a^2}{x^2}\right)}$

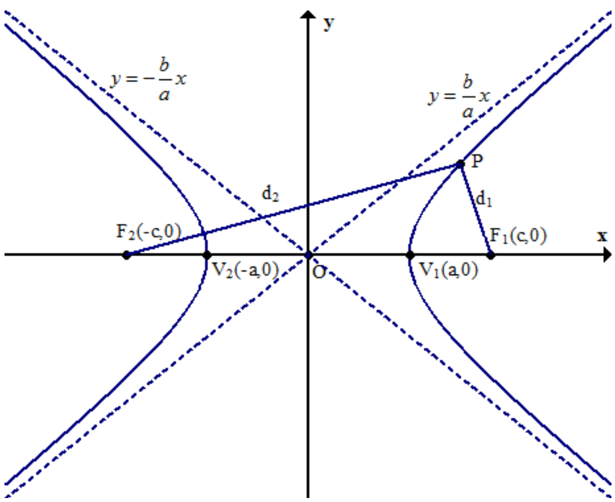
Notice that , if this equation is to have a solution, then $1-\frac{a^2}{x^2}$ must be positive or zero (only then the radical is defined), that is $1\geq\frac{a^2}{x^2}$ or $x^2 \geq a^2$, which means $x \geq a$ or $x \leq -a$. When |x| is large, then $\frac{a^2}{x^2}$ is a small number

($\dfrac{a^2}{x^2} \approx 0$) and therefore $\sqrt{\left(1-\dfrac{a^2}{x^2}\right)} \approx 1$. As a consequence, $y \approx \pm\dfrac{b}{a}x$ when |x| is large. This means that the lines $y = \pm\dfrac{b}{a}x$ are asymptotes for the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. Note that the asymptotes are lines that pass through the origin and have the slopes $\dfrac{b}{a}$, $-\dfrac{b}{a}$

Our findings are summarized below:

Standard equation of a hyperbola with center at (0,0) and **foci on the x-axis**: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, where $b^2 = c^2 - a^2$

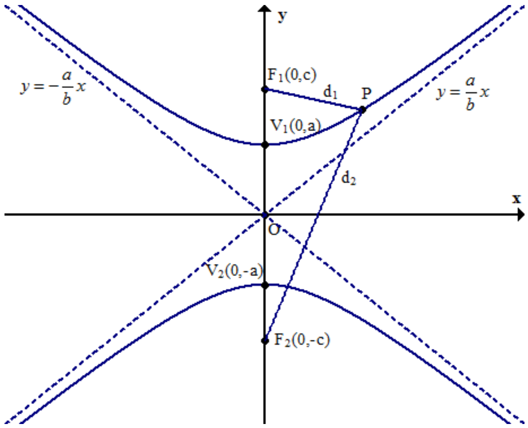
Center: (0,0)
Foci: (c,0), (-c,0);
Vertices: (a,0),(-a,0);
a= dist(vertex, center), c = dist(focus, center) ; c >a
Asymptotes: $y = \dfrac{b}{a}x$ and $y = -\dfrac{b}{a}x$
Graph:



Equation of a hyperbola with the center at the origin and the foci on the y- axis

Work, similar to the above, leads to the following facts:

Standard equation of a hyperbola with center at (0,0) and **foci on the y-axis**: $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$, where $b^2 = c^2 - a^2$
Center: (0,0)
Foci: (0,c), (0,-c)
Vertices: (0,a),(0,-a)
a= dist(vertex, center), c = dist(focus, center); c >a,
Asymptotes: $y = \dfrac{a}{b}x$ and $y = -\dfrac{a}{b}x$
Graph:



Example

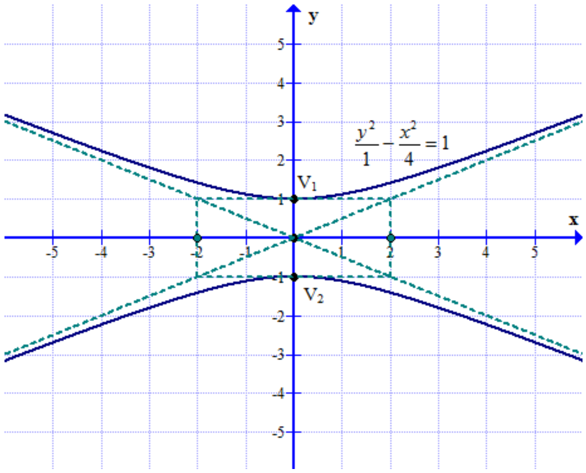
Graph $4y^2 - x^2 = 4$. Find the coordinates of the center, vertices and foci. Write the equations of asymptotes.

1. Write the equation in the standard form (divide each term by 4): $\frac{y^2}{1} - \frac{x^2}{4} = 1$

2. Find the x-intercepts: setting y = 0, we get $-\frac{x^2}{4} = 1$ *or* $x^2 = -4$*; there is no x-intercepts, so* $b^2 = 4$ *and* $b = \sqrt{4} = 2$

2 . Find the y-intercepts: setting x = 0, we get $\frac{y^2}{1} = 1$ *or* $y^2 = 1$ *or* $y = \pm\sqrt{1} = \pm 1$*; there are y intercepts, so* $a^2 = 1$ *and* $a = \sqrt{1} = 1$*. The y-intercepts are the vertices of the hyperbola.*

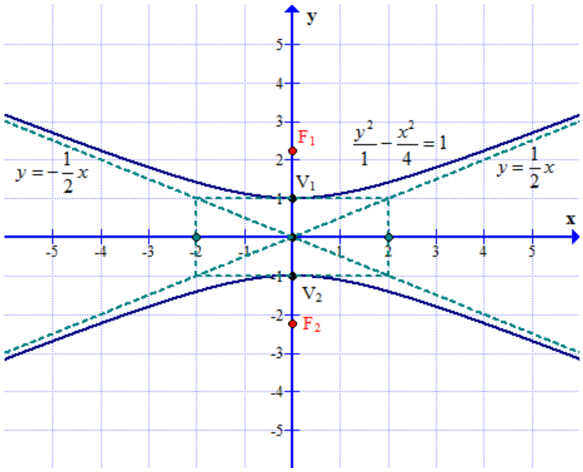
4. Plot $\pm b = \pm 2$ *on the x-axis and* $\pm a = \pm 1$ *on the y-axis and draw a rectangle (using a dashed line) containing these points with the sides parallel to the x- and y- axis. Identify vertices of the hyperbola (y-intercepts)*
5. Draw the diagonals of this rectangle and extend them to form the lines. These are the asymptotes for the hyperbola. The asymptotes divide the plane into 4 parts- the graph of the hyperbola will be in the parts that contain the vertices (intercepts)
6. Draw each branch of the hyperbola, by starting close to one of the asymptotes, going to a vertex and continuing to the other asymptote.



The center is at the origin: (0,0)
Since the vertices of a hyperbola are either x- or y-intercepts, we see that the vertices are: (0,2), (0,-2)(no x-intercepts)

Since $a = 1$, $b = 2$, and $b^2 = c^2 - a^2$, we get $4 = c^2 - 1$ or $c^2 = 5$, or $c = \sqrt{5}$. Therefore the foci, which are on the transverse axis (hence on the y-axis) are at $(0, -\sqrt{5})$ *and* $(0, \sqrt{5})$

The asymptotes are passing through (0,0) and the vertices of the rectangle (points (2,1) and (-2,1), respectively), so their equations are $y = \frac{1}{2}x$ *and* $y = -\frac{1}{2}x$



Example

Find the equation of the hyperbola with vertex at (-3,0) and foci at (±5, 0)

1. Since the foci are (-5,0) and (5,0) , then the center , which is halfway between the foci (the center is the midpoints of the segment with endpoints (-5,0) and (5,0)), ***is at (0,0)****. Thus, c = dist (center, focus) = 5*
2. Since the center is at (0,0), then the second vertex is at (3, 0) and a = dist (center, vertex) = 3.

3. Since the vertices and foci are on the x- axis the equation of the hyperbola is of the form: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

3. Since a = 3 and c = 5, we have

$$b^2 = c^2 - a^2 = 5^2 - 3^2 = 25 - 9 = 16$$

Hence, the equation of the hyperbola with given properties is $\frac{x^2}{9} - \frac{y^2}{16} = 1$

If a hyperbola with center at (0,0) is shifted so its center moves to (h,k), its equation becomes

$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, if the foci and vertices are on the line parallel to the x-axis

and

$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$, if the foci and vertices are on the line parallel to the y-axis

Example

Graph $\frac{(x-3)^2}{4} - \frac{(y+2)^2}{16} = 1$. Find the coordinates of the center, foci and vertices. Find the equations of the asymptotes.

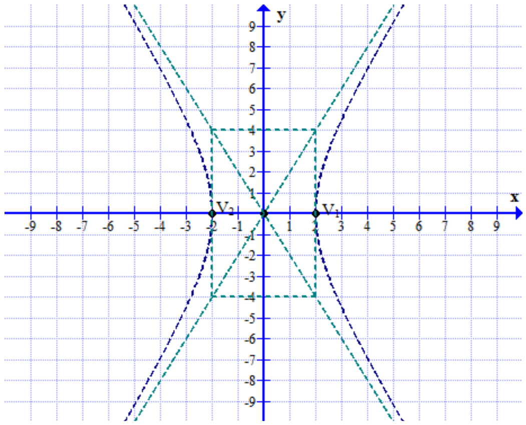
Note that this is the hyperbola $\frac{x^2}{4} - \frac{y^2}{16} = 1$, *shifted, so its center is at (3,-2), hence shifted 3 units to the right and 2 units down.*

1. *Graph, using a dashed line, an hyperbola* $\dfrac{x^2}{4} - \dfrac{y^2}{16} = 1$

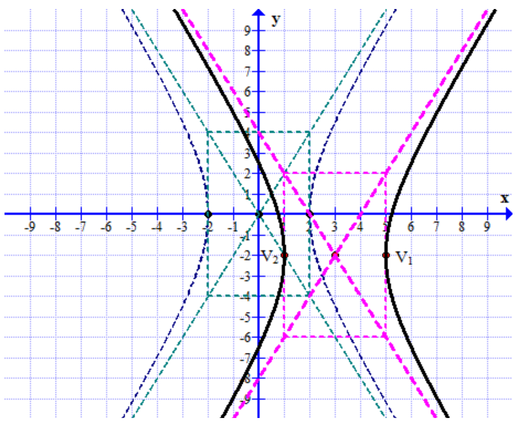
Center at (0,0); x-intercepts: $x = \pm\sqrt{4} = \pm 2$; $a^2 = 4$, $a = 2$; *vertices : (-2,0), (2,0);*

y-intercepts: none ; $b^2 = 16$, $b = 4$; $c^2 = a^2 + b^2 = 20$, $c = \sqrt{20} = 2\sqrt{5}$; *foci:* $(-2\sqrt{5},0)$, $(2\sqrt{5},0)$

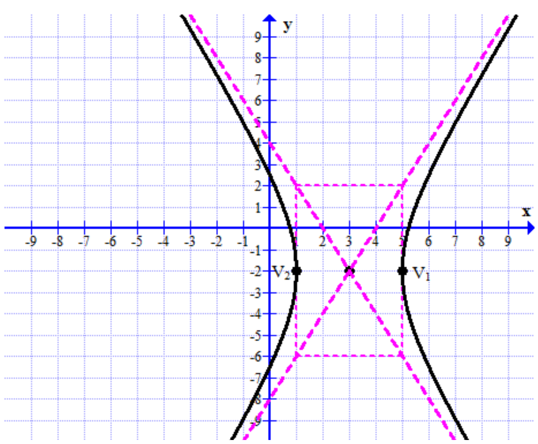
Asymptotes: $y = \pm 2x$



2. *Shift the center, vertices and foci 3 units to the right and 2 units down. Shift the rectangle and the asymptotes. Draw the hyperbola.*



3. *Erasing the original graph gives a clearer picture of the graph*



4. *We can use the graph and the knowledge of transformations to obtain the coordinates of the key points for the hyperbola* $\dfrac{(x-3)^2}{4} - \dfrac{(y+2)^2}{16} = 1$

Center: (3,-2)
Vertices: (1,-2), (5,-2)
Foci: $(3-2\sqrt{5},-2)$ *and* $(3+2\sqrt{5},-2)$
Asymptotes: y +2 = ±2(x-3) or after simplifying: y = 2x-8 and y = -2x+4.

Example
Write the given equation in the standard form. Determine the coordinates of the center, vertices and foci. Find the equations of the asymptotes.

$4x^2 - 9y^2 + 16x - 18y + 43 = 0$

1. *Re-write the equation so the x-terms and the y-terms are together; move the number to the right hand side*
   $(4x^2 + 16x) + (-9y^2 - 18y) = -43$
2. *From the x- group, factor out the coefficient of x^2; from the y-group- factor the coefficient of y^2*

$4(x^2 + 4x) + (-9)(y^2 + 2y) = -43$

3. *Complete the square in each group; add appropriate numbers to the right hand side*

$4(x^2 + 4x + 4) + (-9)(y^2 + 2y + 1) = -43 + 4\cdot 4 + (-9)\cdot 1$

4. *Write each group as a perfect square; add numbers on the right*

$4(x+2)^2 - 9(y + 1)^2 = -36$

5. *Divide each term by the number on the right hand side (-36) and simplify*

$$\frac{4(x+2)^2}{-36}-\frac{9(y+1)^2}{-36}=\frac{-36}{-36}$$
$$-\frac{(x+2)^2}{9}+\frac{(y+1)^2}{4}=1$$

*This is the **standard equation of the hyperbola. Center** is at (-2,-1).*

Hence, this hyperbola is obtained by shifting the hyperbola $-\frac{x^2}{9}+\frac{y^2}{4}=1$ *two units to the left and one unit down. Because the equation* $-\frac{x^2}{9}+\frac{y^2}{4}=1$ *has y intercepts and no x- intercepts (minus is in front of the x-term!), we conclude that $a^2 = 4$ and $b^2 = 9$ and that vertices and foci are on the y-axis. Moreover, $c^2 = a^2 + b^2 = 4+9 = 13$, so c=* $\sqrt{13}$ *. Therefore, for the hyperbola* $-\frac{x^2}{9}+\frac{y^2}{4}=1$, *vertices are (0,-2), (0,2); foci are (0, -*$\sqrt{13}$ *), (0,*$\sqrt{13}$ *) and the asymptotes are* $y=\pm\frac{2}{3}x$.

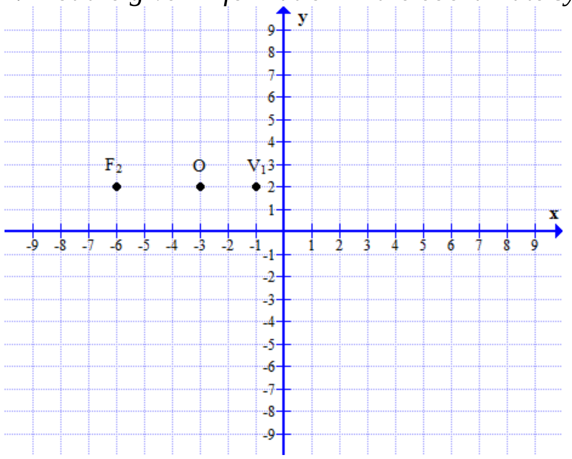
Shifting the hyperbola $-\frac{x^2}{9}+\frac{y^2}{4}=1$ *2 units to the left and 1 unit down, yields the hyperbola*

$$-\frac{(x+2)^2}{9}+\frac{(y+1)^2}{4}=1$$

*for which the **vertices** are (-2, ±2-1) or (-2, 1) and (-2, -3) ;*
*the **foci** are (-2, ±*$\sqrt{13}$ *-1) or (-2,* $\sqrt{13}$ *-1) and (-2,-* $\sqrt{13}$ *-1);*
*the equations of **asymptotes** are* $(y+1)=\pm\frac{2}{3}(x+2)$ *or* $y=\frac{2}{3}x+\frac{1}{3}$ *and* $y=-\frac{2}{3}x-\frac{7}{3}$

Example
Find the equation of the hyperbola with center at (-3,2), focus at (-6,2) and vertex at (-1,2). Graph the hyperbola.

1. Plot the given information in the coordinate system and use them to determine key values for the hyperbola

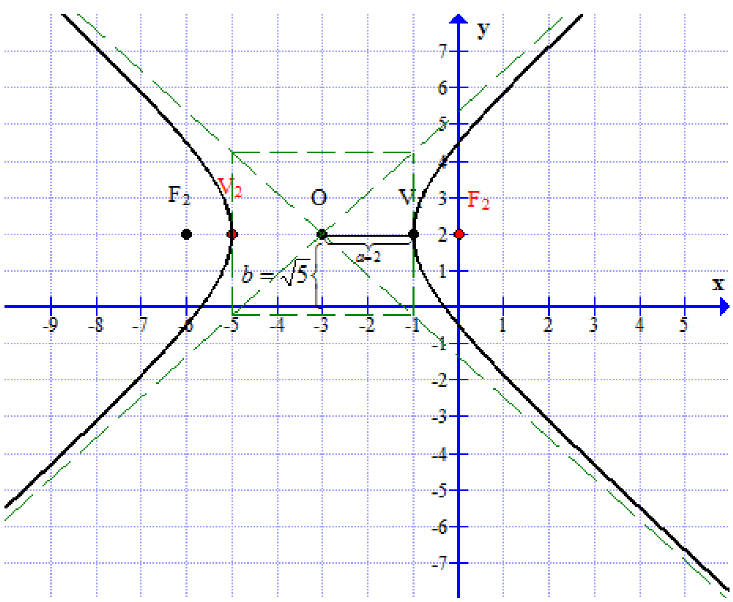


Since c = dist(center, focus), then c = 3 and the second focus is at (0, 2). Since a = dist(center, vertex) , then a = 2 and the second vertex is at (-5,2). Also, $b^2 = c^2 - a^2 = 3^2 - 2^2 = 9-4=5$*, so* $b = \sqrt{5}$*.Vertices and foci are on the line parallel to the y-axis, therefore the standard equation of this hyperbola will be of the form*

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$$

Using values that we determined, the equation is $\frac{(x-(-3))^2}{4} - \frac{(y-2)^2}{5} = 1$ *or* $\frac{(x+3)^2}{4} - \frac{(y-2)^2}{5} = 1.$

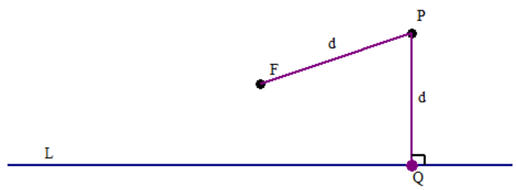
We can use these values to draw the rectangle , asymptotes and the hyperbola itself.



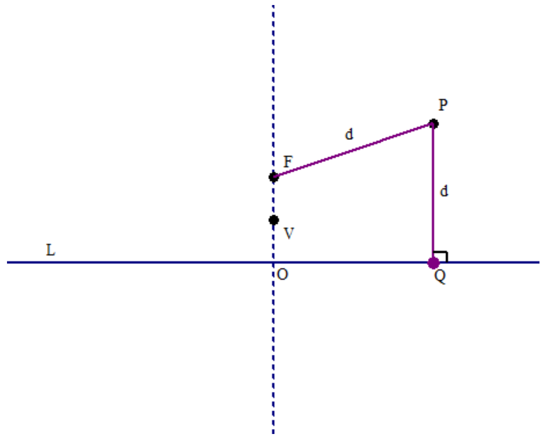
10.3 Parabola

Definition: Given a line L (called the *directrix*) and a point F (called *focus*), a ***parabola*** is the set of points P that are equidistant from the line L and the point P, that is the set of points P such that

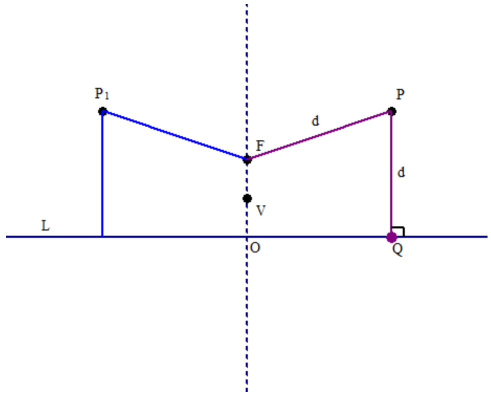
dist (P,F) = dist (P,Line L) = d



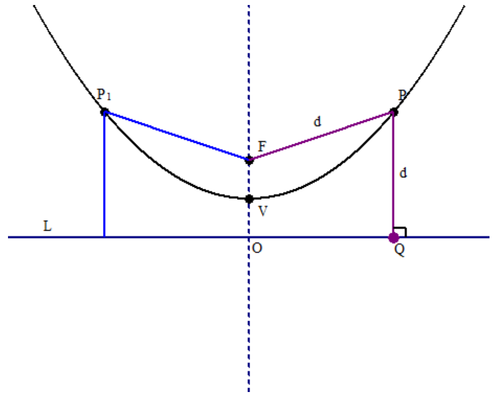
Notice that the point V, which is halfway between the point F and the directrix L is on the parabola, since dist(V,F) = dist (V, Line L). Point V is called the *vertex* of the parabola



Notice also that the point P_1, symmetric to P with respect to the line FV, is also on the parabola as it satisfies dist (P_1,F) = dist (P_1,Line L). Therefore, a parabola is symmetric with respect to the line FV.



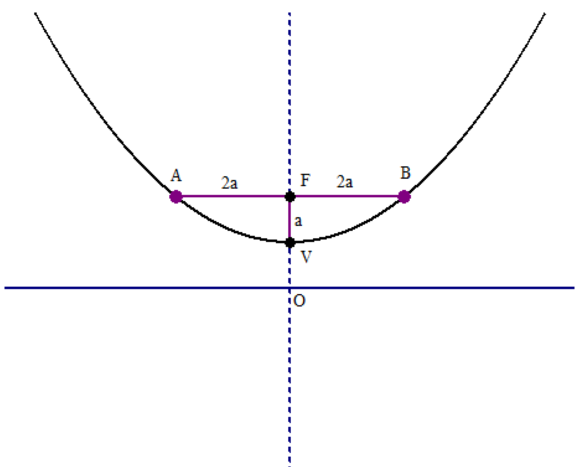
Here is the graph of a parabola.



Let's summarize the properties of a parabola.

- Vertex in on the line perpendicular to the directrix passing through the focus
- Vertex is halfway between the focus and the directrix
- The line through the vertex and the focus is the line of symmetry of the graph
- The line of symmetry is perpendicular to the directrix

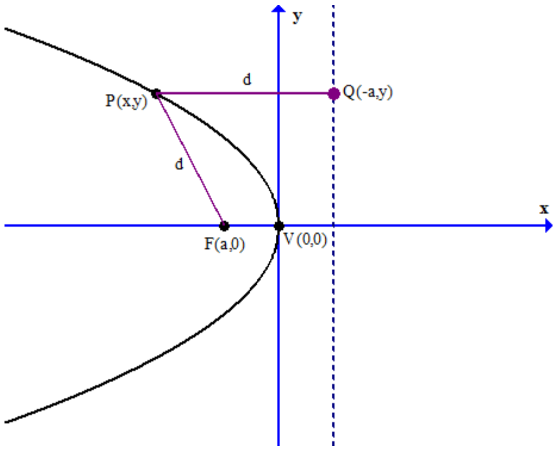
Also notice that the segment perpendicular to the axis of symmetry with one endpoint at the focus F and the other on the parabola has the length that is twice the length between the focus and the directrix: dist(F, A)= dist(F,B) = 2 dist(F, V) = 2a. Segment AB is called *latus rectum* and determines how wide or how narrow the parabola is.



We would like to describe a parabola in algebraic terms, that is by an equation.

Equation of a parabola with vertex at the origin (0,0) and the focus F at (a,0)

First notice that when vertex is at the origin and the focus is at (a,0), then the directrix has the equation $x = -a$. We will use the definition to derive the equation. Let P(x,y) be a point on the parabola.



By definition, dist (P,F)= dis(P, Q). This means that (using the distance formula)

$$\sqrt{(x-a)^2+(y-0)^2}=\sqrt{(x+a)^2+(y-y)^2}$$

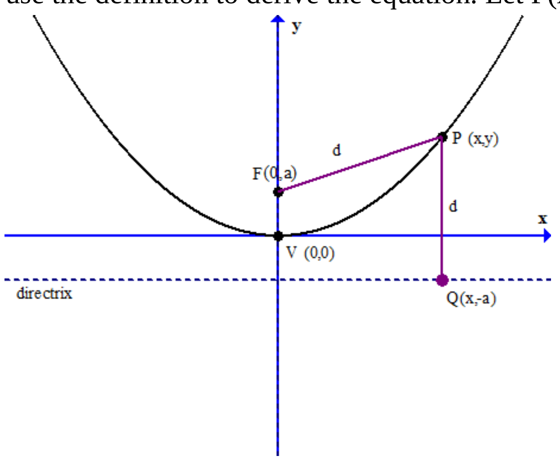
After squaring both sides and expanding the squares we get

$$x^2-2ax+a^2+y^2=x^2+2ax+a^2$$

This equation simplifies to $y^2=4ax$

Equation of a parabola with vertex at the origin (0,0) and the focus F at (0,a)

First notice that when vertex is at the origin and the focus is at (0,a), then the directrix has the equation y = -a. We will use the definition to derive the equation. Let P(x,y) be a point on the parabola.



By definition, dist (P,F)= dis(P, Q). This means that

$$\sqrt{(x-0)^2+(y-a)^2}=\sqrt{(x-x)^2+(y+a)^2}$$

After squaring both sides and expanding the squares we get

$$x^2+y^2-2ay+a^2=y^2+2ay+a^2$$

This equation simplifies to $x^2=4ay$

Remarks:

i)An equation of a parabola contains both x and y; one of these variables comes squared, the other not.

ii) If x- variable is NOT squared, then the x- axis is the axis of symmetry, which means both vertex and the focus are on the x-axis

iii) If y variable is NOT squared then the y-axis is the axis of symmetry, which means that the vertex and focus are on the x-axis.

iv) In both equations ($x^2 = 4ay$ and $y^2 = 4ax$), *a* is the non-zero coordinate of the focus, which means the focus is at (a,0), if the focus is on the x-axis and the focus is at (0,a), when the focus in on the y-axis.
v) If the focus is on the x-axis at (a,0), then the directrix is perpendicular to the x-axis, hence it is a vertical line x = -a. If the focus is on the y-axis at (0,a), then the directrix is perpendicular to the y-axis, hence it is a horizontal line y = -a.
vi) Vertex is always mid-way between the focus and the directrix
vii) The latus rectum segment passes through the focus and is perpendicular to the axis of symmetry. The length of this segment is 4 times the length between focus and the vertex (= 4|a|), with the focus being its midpoint

Example:

Find the equation of the parabola with focus at F(0,2) and vertex at (0,0).
Since focus F is on the y-axis, the equation is of the form (y is NOT squared!) $x^2 = 4ay$ *. Since the non-zero coordinate of the focus is 2, a = 2. Hence the equation is* $x^2 = 8y$.

Example

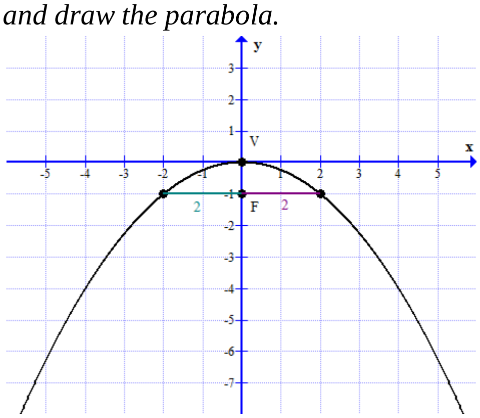
Find the equation of a parabola with the directrix x = - ½ and vertex at (0,0).
Since the directrix is perpendicular to the line of symmetry and it is a vertical line x = -1/2, then the axis of symmetry is the x-axis. Therefore, in the standard form, x will NOT be squared and the equation will be of the form $y^2 = 4ax$*. Since the vertex is half-way between the directrix and focus, the focus is at (1/2,0), hence, a = ½ and the equation is* $y^2 = 2x$

Example

Graph the equation $x^2 = -4y$. Find the coordinates of the focus and the equation of the directrix.

The equation is of the form $x^2 = 4ay$*. Therefore, the vertex is at (0,0) and the axis of symmetry is the y-axis. Focus F has coordinates (0, a), where a satisfies 4a = -4. This means that a = -1, so F=(0, -1).*

Therefore the parabola will open down. To see how wide the parabola is, draw the latus rectum segment. Note first, that the distance between vertex and focus, dis(V,F), is 1 . Starting from the focus F, draw a segment perpendicular to the axis of symmetry (which is the y-axis) of length 2dist(V,F) = 2 to obtain a point on the parabola, then go in the opposite direction the same distance (2) to obtain the second endpoint of latus rectum and draw the parabola.



Parabola with the vertex at (h,k)

First recall that when a graph of a function y = f(x) is moved to the left or right by h units, the equation becomes

$y = f(x \pm h)$, that is the x in the original equation is replaced by $x \pm h$. When we move the graph by k units up or down, the equation becomes $y = f(x) \pm k$, or after moving k to the left hand side : $y \mp k = f(x)$. This means that y in the original equation is replaced by $y \pm k$. If then the graph of $y = f(x)$ is moved h units to the left/right and k units up/down, then the graph has the equation $y \mp k = f(x \pm h)$. Simply put, the x is replaced by $x \pm h$ and the y is replaced by $y \mp k$.

Taking this into account, we can state that when a parabola $y^2 = 4ax$ (or $x^2 = 4ay$) is shifted h units to the left/right and k units up/down, its equation will be $(y \mp k)^2 = 4a(x \pm h)$ (or $(x \pm h)^2 = 4a(y \mp k)$). Vertex of the parabola will be moved accordingly.

Therefore, **the equation of a parabola with vertex at (h,k) and the axis of symmetry parallel to the x-axis is $(y - k)^2 = 4a(x - h)$,**

and the equation of a parabola with vertex at (h,k) and the axis of symmetry parallel to the y-axis is $(x - h)^2 = 4a(y-k)$.

These equations are called **standard equations** of a parabola

Example

Graph $(y + 3)^2 = -8(x-4)$. Find the coordinates of the vertex, focus and the equation of the directrix.

a) Re-write the equation in the standard form: $(y-(-3))^2 = -8(x-4)$

b) Determine the vertex: $V= (h,k) = (4,-3)$

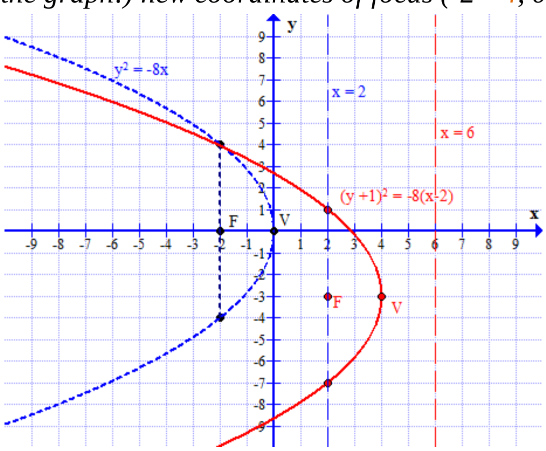
c) Determine the original parabola: The parabola $(y-(-3))^2 = -8(x-4)$ is obtained by shifting the parabola $y^2 = -8x$, so its vertex is at (4,-3) (shifting 4 units to the right and 3 units down)

d) Draw the coordinate system to graph the equation.

e) Draw the parabola $y^2 = -8x$. Use a dashed line.

The axis of symmetry is the x-axis. $-8= 4a$, so $a = -2$ and hence focus $F =(-2,0)$. The directrix has the equation $x = -(-2)$ or $x = 2$

f) Move the parabola, its focus and the directrix 4 units to the right and 3 units down. Determine (read from the graph!) new coordinates of focus $(-2+ 4, 0- 3) = (2, -3)$ and the equation of the directrix : $x-4 = 2$ or $x = 6$.



Example

Graph the equation $x^2 + 4x + 2y - 2 = 0$.

The graph of this equation is a parabola (x variable is squared and y is not). It is not in the standard form.

1) Write the equation in the standard form

$$x^2 + 4x + 2y - 2 = 0$$

a) Leave the terms with x on the left hand side and move all remaining terms to the right

$$x^2 + 4x = - 2y + 2$$

b) Complete the square on the left hand side ($4/2 = 2$, $2^2 = 4$) and write it as a square; simplify the right hand side

$x^2 + 4x + 4 = - 2y + 2 + 4$

$(x+2)^2 = - 2y + 6$

c) Factor out the coefficient of y on the right hand side

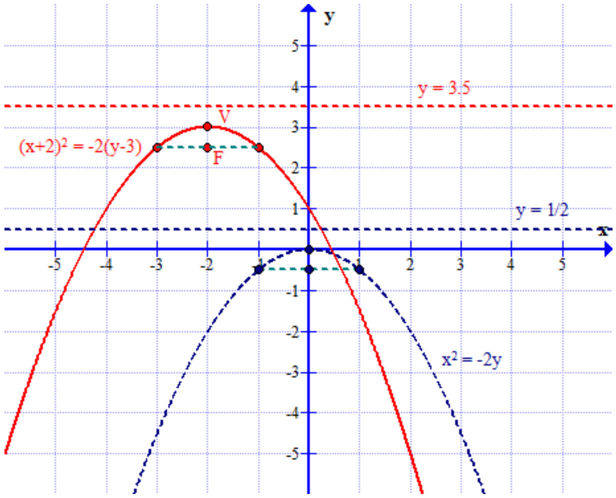
$(x+2)^2 = -2(y - 3)$

2. Determine the vertex of the parabola

Equation: $(x-(-2))^2 = -2(y-3)$, so the vertex is V= (-2, 3)

3. Graph, using a dashed line, the parabola $x^2 = -2y$ and shift it 2 units to the left and 3 unit up.

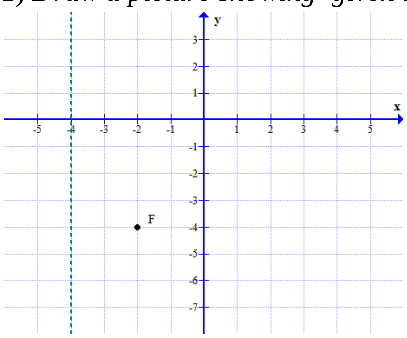
Parabola $x^2 = -2y$: vertex at (0,0); axis of symmetry: y-axis; focus: -2 = 4a, a = -½; focus at (0,-1/2)



Example

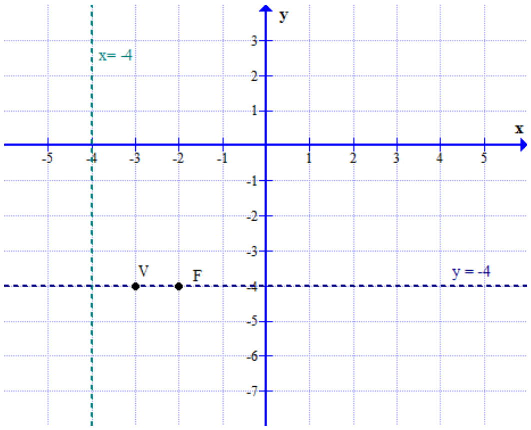
Find the equation of the parabola with focus at (-2,-4) and the directrix x = -4.

1) *Draw a picture showing given information*



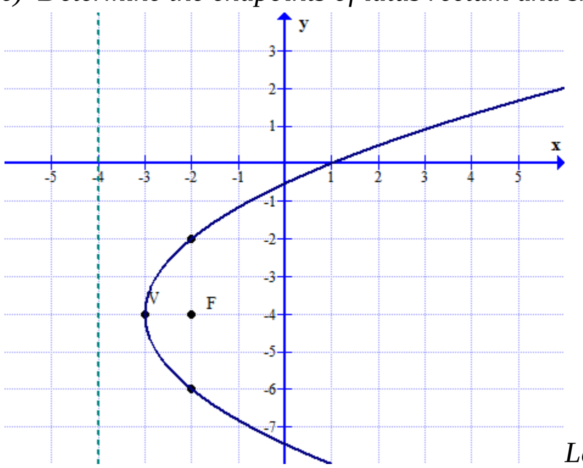
b) Find the vertex

The line of symmetry (perpendicular to the directrix and passing through focus F) is the line $y = -4$. Since vertex is half-way between the focus and the directrix and lies on the axis of symmetry, the vertex is at (-3,-4)



V = (-3,-4)

c) Determine the endpoints of latus rectum and sketch the parabola



Latus rectum endpoints: (-2,-6), (-2,-2)

d) Determine the equation

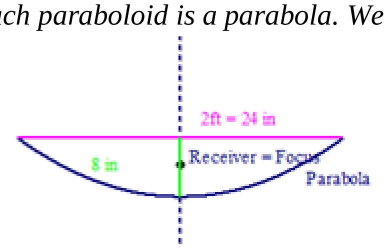
The axis of symmetry is parallel to the x-axis (x is NOT squared); Vertex is at(-3,-4); Focus is to the right of vertex and dist(V,F) = 1, so a = 1.

Equation: $(y + 4)^2 = 4(1)(x+3)$ *or* $(y + 4)^2 = 4(x+3)$.

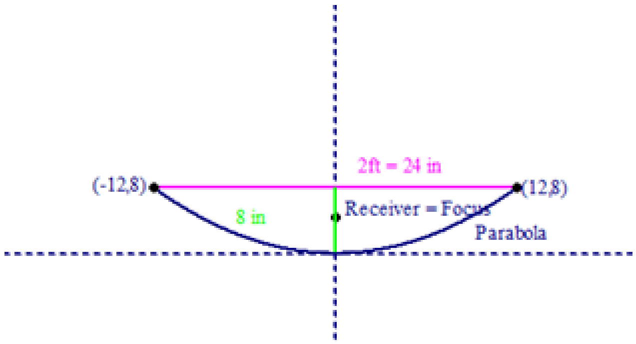
Example

A cable TV receiving dish is in the shape of paraboloid of revolution. Find the location of the receiver which is placed at the focus, if the dish is 2 feet across at the opening and 8 inches deep.

A paraboloid of revolution is obtained by revolving a parabola about its axis of symmetry. The cross section of such paraboloid is a parabola. We can draw a picture as below.



We can introduce the coordinate system with the origin at the vertex of the parabola and find the equation of that parabola.



The standard equation of such parabola is $x^2 = 4ay$. *The given information allows us to determine two points on this parabola: (12,8) and (-12,8). Using the fact that the point (12,8) is on the parabola (so* $x = 12$ *and* $y = 8$ *satisfy the equation), we can find value of a:*

$$12^2 = 4a(8)$$

$a = 144/32 = 9/2 = 4.5$ *and therefore the focus is at (0,4.5)*

Therefore, the focus should be 4.5 inches from the vertex of the paraboloid.